Dr Oliver Mathematics
Extended Mathematics Certificate
Sample Assessment Materials: Non-Calculator
1 hour 15 minutes

The total number of marks available is 60.
You must write down all the stages in your working.

1.

$$f(x) = 4x + 6.$$

(a) Find $f(-3)$. (1)

Solution

$$\begin{aligned} f(-3) &= 4(-3) + 6 \\ &= -12 + 6 \\ &= \underline{\underline{-6}}. \end{aligned}$$

(b) Find an equation for the line perpendicular to (2)

$$y = 4x + 6$$

that passes through the point $(0, -8)$.

Solution

Well,

$$m_{\text{normal}} = -\tfrac{1}{4}$$

and an equation for the line is

$$y + 8 = -\tfrac{1}{4}(x - 0) \Rightarrow \underline{\underline{y = -\tfrac{1}{4}x - 8}}.$$

Point A with coordinates $(a, 10)$ and point B with coordinates $(3, b)$ both lie on

$$y = 4x + 6.$$

(c) Find the length of AB. (3)
Give you answer in the form $c\sqrt{d}$, where c and d are integers.

Solution

Now,

$$\begin{aligned} y = 10 \Rightarrow 4a + 6 &= 10 \\ \Rightarrow 4a &= 4 \\ \Rightarrow a &= 1 \end{aligned}$$

and

$$\begin{aligned} x = 3 \Rightarrow y &= 4(3) + 6 \\ \Rightarrow y &= 12 + 6 \\ \Rightarrow y &= 18; \end{aligned}$$

so, $A(1, 10)$ and $B(3, 18)$.

Finally,

$$\begin{aligned} AB &= \sqrt{(3-1)^2 + (18-10)^2} \\ &= \sqrt{2^2 + 8^2} \\ &= \sqrt{4 + 64} \\ &= \sqrt{68} \\ &= \sqrt{4 \times 17} \\ &= \sqrt{4} \times \sqrt{17} \\ &= \underline{\underline{2\sqrt{17}}}; \end{aligned}$$

hence, $\underline{\underline{c = 2}}$ and $\underline{\underline{d = 17}}$.

2. (a) Simplify (1)

$$\sqrt{18}.$$

Solution

$$\begin{aligned} \sqrt{18} &= \sqrt{9 \times 2} \\ &= \sqrt{9} \times \sqrt{2} \\ &= \underline{\underline{3\sqrt{2}}}. \end{aligned}$$

(b) Simplify (2)

$$\sqrt{8} + \sqrt{18} - 3.$$

Solution

Now,

$$\begin{aligned}\sqrt{8} &= \sqrt{4 \times 2} \\ &= \sqrt{4} \times \sqrt{2} \\ &= 2\sqrt{2}\end{aligned}$$

and

$$\begin{aligned}\sqrt{8} + \sqrt{18} - 3 &= 2\sqrt{2} + 3\sqrt{2} - 3 \\ &= \underline{\underline{5\sqrt{2} - 3}}.\end{aligned}$$

$$\frac{\sqrt{2} + 6}{\sqrt{8} + \sqrt{18} - 3}.$$

(c) Hence write in the form (4)

$$\frac{a\sqrt{b} + c}{d},$$

where a, b, c, and d are integers.

Solution

$$\begin{aligned}\frac{\sqrt{2} + 6}{\sqrt{8} + \sqrt{18} - 3} &= \frac{\sqrt{2} + 6}{5\sqrt{2} - 3} \\ &= \frac{\sqrt{2} + 6}{5\sqrt{2} - 3} \times \frac{5\sqrt{2} + 3}{5\sqrt{2} + 3}\end{aligned}$$

$\times$	$\sqrt{2}$	$+6$
$5\sqrt{2}$	10	$+30\sqrt{2}$
$+3$	$+3\sqrt{2}$	$+18$

$\times$	$5\sqrt{2}$	-3
$5\sqrt{2}$	50	$-15\sqrt{2}$
$+3$	$+15\sqrt{2}$	-9

$$= \frac{33\sqrt{2} + 28}{50 - 9}$$
$$= \underline{\underline{\frac{33\sqrt{2} + 28}{41}}};$$

hence, $\underline{\underline{a = 33}}$, $\underline{\underline{b = 2}}$, $\underline{\underline{c = 28}}$, and $\underline{\underline{d = 41}}$.

3. Here are the first few rows of Pascal's Triangle.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

(a) Using this information, expand (2)
$$(e + f)^3.$$

Solution
$$(e + f)^3 = \underline{\underline{e^3 + 3e^2 f + 3ef^2 + f^3}}.$$

Given that
$$(e + f)^4 = e^4 + 4e^3 f + 6e^2 f^2 + 4ef^3 + f^4,$$

(b) (i) work out (2)
$$7^4 + 12 \times 7^3 + 6 \times 7^2 \times 3^2 + 28 \times 3^3 + 3^4.$$

Solution
Well,

$$\begin{aligned} & 7^4 + 12 \times 7^3 + 6 \times 7^2 \times 3^2 + 28 \times 3^3 + 3^4 \\ = \quad & (7)^4 + 4(7)^3(4) + 6(7)^2(3)^2 + 4(7)(3)^3 + (3)^4 \\ = \quad & (7+3)^4 \\ = \quad & 10^4 \\ = \quad & \underline{\underline{10\,000}}. \end{aligned}$$

(ii) expand and simplify (3)

$$(2e+f)^4.$$

Solution
Now,

$$\begin{aligned} (2e+f)^4 &= (2e)^4 + 4(2e)^3 f + 6(2e)^2 f^2 + 4(2e) f^3 + f^4 \\ &= \underline{\underline{16e^4 + 32e^3 f + 24e^2 f^2 + 8ef^3 + f^4}}. \end{aligned}$$

4. (a) (i) Simplify (1)

$$81^{\frac{3}{4}}.$$

Solution

$$\begin{aligned} 81^{\frac{3}{4}} &= (81^{\frac{1}{4}})^3 \\ &= 3^3 \\ &= \underline{\underline{27}}. \end{aligned}$$

(ii) Write (1)

$$\frac{1}{9^2}$$

in the form 3^n.

Solution

$$\frac{1}{9^2} = \frac{1}{(3^2)^2} = \frac{1}{3^4} = \underline{\underline{3^{-4}}};$$

hence, $\underline{\underline{n = -4}}$.

$$27^{-\frac{2}{3}} \times 3^{2y+1} \times \frac{1}{9^2} \times 81^{\frac{3}{4}} = 27.$$

(b) Find the value of y. (4)

Solution

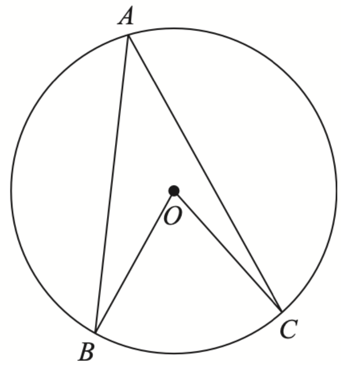
Well,

$$27^{-\frac{2}{3}} = (27^{\frac{1}{3}})^{-2} = 3^{-2}$$

and

$$\begin{aligned} & 27^{-\frac{2}{3}} \times 3^{2y+1} \times \frac{1}{9^2} \times 81^{\frac{3}{4}} = 27 \\ \Rightarrow\ & 3^{-2} \times 3^{2y+1} \times 3^{-4} \times 27 = 27 \\ \Rightarrow\ & 3^{-2} \times 3^{2y+1} \times 3^{-4} = 1 \\ \Rightarrow\ & 3^{-2+2y+1-4} = 1 \\ \Rightarrow\ & 3^{2y-5} = 1 \\ \Rightarrow\ & 2y - 5 = 0 \\ \Rightarrow\ & 2y = 5 \\ \Rightarrow\ & \underline{\underline{y = 2\tfrac{1}{2}}}. \end{aligned}$$

5. The diagram shows a circle, centre O. (4)
A, B, and C are points on the circumference of the circle.



Prove that the angle subtended by the arc at the centre is twice the angle subtended at the circumference.

Solution

Let $x = \angle BAO$.
Then $\angle OAB = x$ (base angles)
$\angle AOB = 180 - x - x = 180 - 2x$ (completing the triangle).

Let $y = \angle CAO$.
Then $\angle OAC = y$ (base angles)
$\angle AOC = 180 - y - y = 180 - 2y$ (completing the triangle).

Now,

$$\begin{aligned}\angle BOC &= 360 - \angle AOB - \angle AOC \\ &= 360 - (180 - 2x) - (180 - 2y) \\ &= 2x + 2y \\ &= 2(x + y) \\ &= 2\angle BAC;\end{aligned}$$

hence, the angle subtended by the arc at the centre is twice the angle subtended at the circumference.

6. The point Q with coordinates $(-2, 0)$ is on the curve $\mathrm{f}(x)$.

   The transformation

$$\mathrm{f}(x + a) + b$$

   of the curve $\mathrm{f}(x)$ moves the point P from $(0, 0)$ to $(3, 4)$.

(a) Write down the coordinates of Q after the transformation (1)

$$f(x + a) + b.$$

Solution
$\underline{\underline{(1, 4)}}$.

(b) Work out the value of a and the value of b. (2)

Solution
$\underline{\underline{a = -3}}$ and $\underline{\underline{b = 4}}$.

The transformation

$$k\,g(dx) + 1$$

of the curve $g(x)$ moves the point R, from $(-3, 2)$ to $(-6, 7)$.

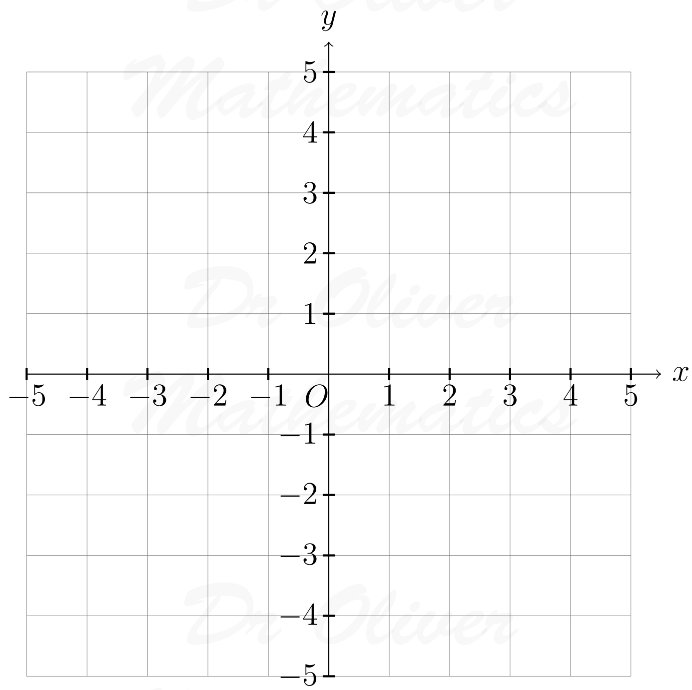
(c) Work out the value of d and the value of k. (3)

Solution
Well,

- we squash with a factor of d horizontally — $\underline{\underline{d = \frac{1}{2}}}$;
- we stretch with a factor of k vertically — $\underline{\underline{k = 3}}$;, and
- add 1.

7. A circle **C** has centre $(0, -3)$ and circumference 4π.

(a) Sketch the graph of **C**. (2)

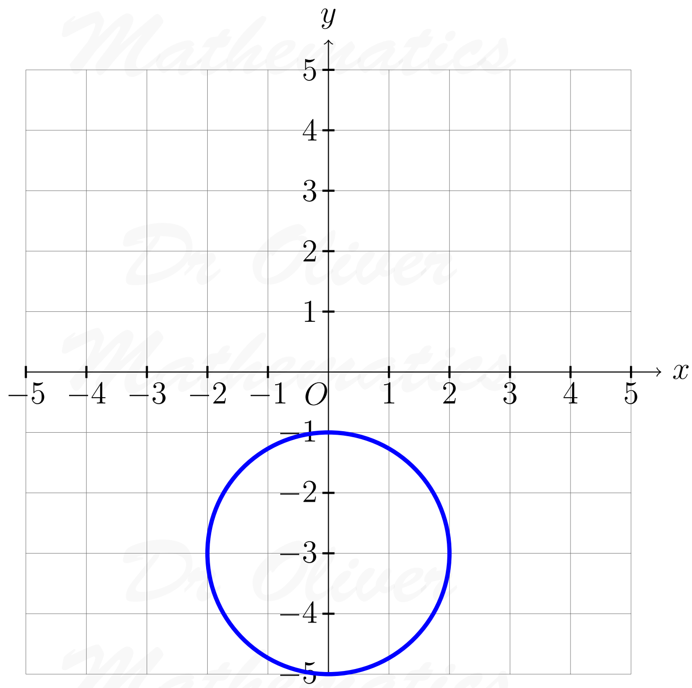


Solution

Well,

$$\begin{aligned}\text{circumference} = 2\pi r \Rightarrow 4\pi &= 2\pi r\\ \Rightarrow r &= 2\end{aligned}$$

and so



The line **L** has equation

$$2x - y = 5.$$

(b) Find, algebraically, the coordinates of the points of intersection of **C** and **L**. (5)

Solution

The circle **C** has equation

$$x^2 + (y+3)^2 = 2^2 \quad (1)$$

and

$$2x - y = 5 \Rightarrow y = 2x - 5 \quad (2).$$

Now,

$$x^2 + [(2x-5)+3]^2 = 2^2 \Rightarrow x^2 + (2x-2)^2 = 4$$

×	$2x$	-2
$2x$	$4x^2$	$-4x$
-2	$-4x$	$+4$

$$\begin{aligned}
&\Rightarrow x^2 + (4x^2 - 8x + 4) = 4 \\
&\Rightarrow 5x^2 - 8x = 0 \\
&\Rightarrow x(5x - 8) = 0 \\
&\Rightarrow x = 0 \text{ or } 5x - 8 = 0 \\
&\Rightarrow x = 0 \text{ or } x = 1\tfrac{3}{5} \\
&\Rightarrow y = -5 \text{ or } y = -1\tfrac{4}{5};
\end{aligned}$$

hence, the coordinates are $\underline{\underline{(0, -5)}}$ and $\underline{\underline{(1\frac{3}{5}, -1\frac{4}{5})}}$.

8. Alex is standing on a tower and throws a ball to Chris who is standing on the ground.

The motion of the ball is modelled by the equation

$$s = -5t^2 + 20t + 7,$$

where s is the height of the ball above the ground, in metres, and t is the time, in seconds, from when Alex throws the ball.

(a) Write down the initial height of the ball? (1)

Solution

$$t = 0 \Rightarrow \underline{\underline{s = 7 \text{ m}}}.$$

(b) Explain why the model is not valid when $t = 5$. (1)

Solution

Well,

$$\begin{aligned}
t = 5 &\Rightarrow s = -5(5^2) + 20(5) + 7 \\
&\Rightarrow s = -125 + 100 + 7 \\
&\Rightarrow s = -18;
\end{aligned}$$

<u>we cannot have a negative height</u>.

(c) Work out the maximum height the ball reaches. (3)

Solution

Now,

$$s = -5t^2 + 20t + 7 \Rightarrow v = -10t + 20$$

and

$$\begin{aligned} v = 0 &\Rightarrow -10t + 20 = 0 \\ &\Rightarrow 10t = 20 \\ &\Rightarrow t = 20 \\ &\Rightarrow \underline{\underline{s = 27 \text{ m}}}. \end{aligned}$$

Chris catches the ball when it is 2 metres above the ground.

(d) Work out the total amount of time the ball is in flight. (4)
Give your answer in the form $a + \sqrt{b}$, where a and b are integers.

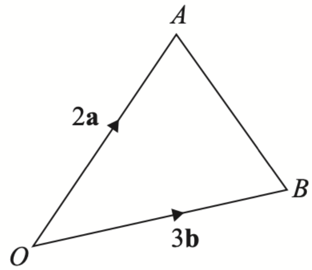
Solution

Well,

$$\begin{aligned} s = 2 &\Rightarrow 2 = -5t^2 + 20t + 7 \\ &\Rightarrow 5t^2 - 20t - 5 = 0 \\ &\Rightarrow 5(t^2 - 4t - 1) = 0 \\ &\Rightarrow t^2 - 4t = 1 \\ &\Rightarrow t^2 - 4t + 4 = 1 + 4 \\ &\Rightarrow (t-2)^2 = 5 \\ &\Rightarrow t - 2 = \pm\sqrt{5} \\ &\Rightarrow t = 2 \pm \sqrt{5}. \end{aligned}$$

But $t \geqslant 0$ (why?) so $\underline{\underline{t = 2 + \sqrt{5}}}$.

9. Here is a picture. (8)



- $\overrightarrow{OA} = 2\mathbf{a}$.
- $\overrightarrow{OB} = 3\mathbf{b}$.
- C is a point such that $\overrightarrow{AC} = \frac{5}{3}\overrightarrow{AB}$.
- D is a point such that $\overrightarrow{AD} = x\mathbf{a} + y\mathbf{b}$ and $\overrightarrow{CD} = -\frac{2}{3}x\mathbf{a} + \frac{13}{33}y\mathbf{b}$.

Find the ratio $OB : BD$.
Give your ratio in its simplest form.

Solution

Well,

$$\begin{aligned} \overrightarrow{AC} &= \tfrac{5}{3}\overrightarrow{AB} \\ &= \tfrac{5}{3}(\overrightarrow{AO} + \overrightarrow{OB}) \\ &= \tfrac{5}{3}(-\overrightarrow{OA} + \overrightarrow{OB}) \\ &= \tfrac{5}{3}(-2\mathbf{a} + 3\mathbf{b}) \\ &= -\tfrac{10}{3}\mathbf{a} + 5\mathbf{b} \end{aligned}$$

and

$$\begin{aligned} \overrightarrow{AD} &= \overrightarrow{AD} + \overrightarrow{CD} \\ &= (-\tfrac{10}{3}\mathbf{a} + 5\mathbf{b}) + (-\tfrac{2}{3}x\mathbf{a} + \tfrac{13}{33}y\mathbf{b}) \\ &= (-\tfrac{10}{3} - \tfrac{2}{3}x)\mathbf{a} + (5 + \tfrac{13}{33}y)\mathbf{b}. \end{aligned}$$

Now, we have two different ways of writing down AD:

$$x\mathbf{a} + y\mathbf{b} \text{ and } (-\tfrac{10}{3} - \tfrac{2}{3}x)\mathbf{a} + (5 + \tfrac{13}{33}y)\mathbf{b}.$$

Look at the xs:

$$\begin{aligned} x = -\tfrac{10}{3} - \tfrac{2}{3}x &\Rightarrow \tfrac{5}{3}x = -\tfrac{10}{3} \\ &\Rightarrow x = -2. \end{aligned}$$

Look at the ys:

$$\begin{aligned} y = 5 + \tfrac{13}{33}y &\Rightarrow \tfrac{20}{33}y = 5 \\ &\Rightarrow \tfrac{1}{33}y = \tfrac{1}{4} \\ &\Rightarrow y = \tfrac{33}{4}. \end{aligned}$$

Next,

$$\begin{aligned} \overrightarrow{BD} &= \overrightarrow{BO} + \overrightarrow{OA} + \overrightarrow{AD} \\ &= -\overrightarrow{OB} + \overrightarrow{OA} + \overrightarrow{AD} \\ &= -3\mathbf{b} + 2\mathbf{a} + (-2\mathbf{a} + \tfrac{33}{4}\mathbf{b}) \\ &= \tfrac{21}{4}\mathbf{b}. \end{aligned}$$

Finally,

$$\begin{aligned} \overrightarrow{OB} : \overrightarrow{BD} &= 3\mathbf{b} : \tfrac{21}{4}\mathbf{b} \\ &= \mathbf{b} : \tfrac{7}{4}\mathbf{b} \\ &= 4\mathbf{b} : 7\mathbf{b} \end{aligned}$$

and, finally,

$$OB : BD = \underline{\underline{\underline{4 : 7}}}.$$