Dr Oliver Mathematics
Mathematics: Advanced Higher
2009 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. (a) Given (3)

$$f(x) = (x+1)(x-2)^3,$$

obtain the values of x for which $f'(x) = 0$

   (b) Calculate the gradient of the curve defined by (4)

$$\frac{x^2}{y} + x = y - 5$$

at the point $(3, -1)$.

2. Given the matrix

$$\mathbf{A} = \begin{pmatrix} t+4 & 3t \\ 3 & 5 \end{pmatrix},$$

   (a) find $\mathbf{A}^{-1}$ in terms of t when $\mathbf{A}$ is non-singular. (3)

   (b) Write down the value of t such that $\mathbf{A}$ is singular. (1)

   (c) Given that the transpose of $\mathbf{A}$ is (1)

$$\begin{pmatrix} 6 & 3 \\ 6 & 5 \end{pmatrix},$$

find t.

3. Given that (4)

$$x^2 e^y \frac{dy}{dx} = 1,$$

and $y = 0$ when $x = 1$, find y in terms of x.

4. Prove by induction that, for all positive integers n, (5)

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}.$$

5. Show that (4)

$$\int_{\ln \frac{3}{2}}^{\ln 2} \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{\mathrm{e}^x - \mathrm{e}^{-x}} \, \mathrm{d}x = \ln\left(\frac{9}{5}\right).$$

6. (a) Express (3)

$$z = \frac{(1+2\mathrm{i})^2}{7-\mathrm{i}}$$

in the form $a + \mathrm{i}b$ where a and b are real numbers.

(b) Show z on an Argand diagram and evaluate $|z|$ and $\arg(z)$. (3)

7. Use the substitution $x = 2\sin\theta$ to obtain the exact value of (6)

$$\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}} \, \mathrm{d}x.$$

8. (a) Write down the binomial expansion of $(1+x)^5$. (1)

(b) Hence show that $(0.9)^5$ is 0.590 49. (2)

9. Use integration by parts to obtain the exact value of (5)

$$\int_0^1 x \tan^{-1} x^2 \, \mathrm{d}x.$$

10. Use the Euclidean algorithm to obtain the greatest common divisor of 1 326 and 14 654, expressing it in the form (4)

$$1\,326a + 14\,654b,$$

where a and b are integers.

11. The curve (5)

$$y = x^{2x^2+1}$$

is defined for $x > 0$.

Obtain the values of y and $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point where $x = 1$.

12. (a) The first two terms of a geometric sequence are $a_1 = p$ and $a_2 = p^2$. Obtain expressions for S_n and S_{2n} in terms of p, where (2)

$$S_k = \sum_{j=1}^{k} a_j.$$

(b) Given that (2)

$$S_{2n} = 65S_n$$

show that $p^n = 64$.

(c) Given also that $a_3 = 2p$ and that $p > 0$, obtain the exact value of p and hence the value of n. (2)

13. The function $\mathrm{f}(x)$ is defined by

$$\mathrm{f}(x) = \frac{x^2 + 2x}{x^2 - 1},\ x \neq \pm 1.$$

(a) Obtain equations for the asymptotes of the graph of $\mathrm{f}(x)$. (3)

(b) Show that $\mathrm{f}(x)$ is a strictly decreasing function. (3)

(c) Find the coordinates of the points where the graph of $\mathrm{f}(x)$ crosses

(i) the x-axis, and (1)

(ii) the horizontal asymptote. (1)

(d) Sketch the graph of $\mathrm{f}(x)$, showing clearly all relevant features. (2)

14. (a) Express (4)

$$\frac{x^2 + 6x - 4}{(x+2)^2(x-4)}$$

in partial fractions.

(b) Hence, or otherwise, obtain the first three non-zero terms in the Maclaurin expansion of (5)

$$\frac{x^2 + 6x - 4}{(x+2)^2(x-4)}.$$

15. (a) Solve the differential equation (6)

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = (x+1)^4,$$

given that $y = 16$ when $x = 1$, expressing the answer in the form $y = \mathrm{f}(x)$.

(b) Hence, find the area enclosed by the graphs of $y = \mathrm{f}(x)$, $y = (1-x)^4$, and the x-axis. (4)

16. (a) Use Gaussian elimination to solve the following system of equations: (5)

$$\begin{aligned} x + y - z &= 6 \\ 2x - 3y + 2z &= 2 \\ -5x + 2y - 4z &= 1. \end{aligned}$$

(b) Show that the line of intersection, L, of the planes $x+y-z=6$ and $2x-3y+2z=2$ has parametric equations (2)

$$\begin{aligned} x &= \lambda \\ y &= 4\lambda - 14 \\ z &= 5\lambda - 20. \end{aligned}$$

(c) Find the acute angle between line L and the plane $-5x+2y-4z=1$. (4)