Dr Oliver Mathematics
Advance Level Mathematics
Core Mathematics 3: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1. Given $y = 2x(3x-1)^5$,

   (a) find $\frac{\mathrm{d}y}{\mathrm{d}x}$, giving your answer as a single fully factorised expression. (4)

   Solution

   $$u = 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2$$
   $$v = (3x-1)^5 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 15(3x-1)^4.$$

   Now,

   $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x \times 15(3x-1)^4 + 2 \times (3x-1)^5 \\ &= 2(3x-1)^4[15x + (3x-1)] \\ &= \underline{\underline{2(3x-1)^4(18x-1)}}.\end{aligned}$$

   (b) Hence find the set of values of x for which $\frac{\mathrm{d}y}{\mathrm{d}x} \leqslant 0$. (2)

   Solution

   $$\frac{\mathrm{d}y}{\mathrm{d}x} \leqslant 0 \Rightarrow \underline{\underline{x \leqslant \tfrac{1}{18} \text{ or } x = \tfrac{1}{3}}}.$$

2. The function f is defined by

   $$\mathrm{f}(x) = \frac{6}{2x+5} + \frac{2}{2x-5} + \frac{60}{4x^2-25},\ x > 4.$$

   (a) Show that (4)

   $$\mathrm{f}(x) = \frac{A}{Bx+C},$$

   where A, B, and C are constants to be found.

Solution

$$\begin{aligned} &\frac{6}{2x+5} + \frac{2}{2x-5} + \frac{60}{4x^2-25} \\ &\equiv \frac{6(2x-5)+2(2x+5)+60}{(2x-5)(2x+5)} \\ &\equiv \frac{16x+40}{(2x-5)(2x+5)} \\ &\equiv \frac{8(2x+5)}{(2x-5)(2x+5)} \\ &\equiv \underline{\underline{\frac{8}{2x-5}}}; \end{aligned}$$

hence, $\underline{\underline{A = 8}}$, $\underline{\underline{B = 2}}$, and $\underline{\underline{C = -5}}$.

(b) Find $\mathrm{f}^{-1}(x)$ and state its domain. (3)

Solution

$$\begin{aligned} y = \frac{8}{2x-5} &\Rightarrow 2x - 5 = \frac{8}{y} \\ &\Rightarrow 2x = \frac{8}{y} + 5 \\ &\Rightarrow 2x = \frac{8+5y}{y} \\ &\Rightarrow x = \frac{8+5y}{2y}; \end{aligned}$$

hence,

$$\underline{\underline{\mathrm{f}^{-1}(x) = \frac{8+5x}{2x}}}$$

and its domain is

$$\underline{\underline{0 < x < 2\tfrac{2}{3}}}.$$

3. The value of a car is modelled by the formula

$$V = 16\,000\mathrm{e}^{-kt} + A,\ t \geqslant 0,\ t \in \mathbb{R},$$

where V is the value of the car in pounds, t is the age of the car in years, and k and A are positive constants.

Given that the value of the car is £17 500 when new and £13 500 two years later,

(a) find the value of A, (1)

Solution

$$t = 0 \Rightarrow 16\,000 + A = 17\,500$$
$$\Rightarrow \underline{\underline{A = 1\,500}}.$$

(b) show that (4)

$$k = \ln\left(\frac{2}{\sqrt{3}}\right).$$

Solution

$$\begin{aligned}16\,000\mathrm{e}^{-2k} + 1\,500 = 13\,500 &\Rightarrow 16\,000\mathrm{e}^{-2k} = 12\,000 \\ &\Rightarrow \mathrm{e}^{-2k} = \tfrac{3}{4} \\ &\Rightarrow \mathrm{e}^{2k} = \tfrac{4}{3} \\ &\Rightarrow 2k = \ln\tfrac{4}{3} \\ &\Rightarrow k = \tfrac{1}{2}\ln\tfrac{4}{3} \\ &\Rightarrow k = \ln\left[\left(\frac{4}{3}\right)^{\frac{1}{2}}\right] \\ &\Rightarrow \underline{\underline{k = \ln\left(\frac{2}{\sqrt{3}}\right)}},\end{aligned}$$

as required.

(c) Find the age of the car, in years, when the value of the car is £6 000. (4)
Give your answer to 2 decimal places.

Solution

$$\begin{aligned}16\,000\mathrm{e}^{-kt} + 1\,500 = 6\,000 &\Rightarrow 16\,000\mathrm{e}^{-kt} = 4\,500\\ &\Rightarrow \mathrm{e}^{-kt} = \tfrac{9}{32}\\ &\Rightarrow \mathrm{e}^{kt} = \tfrac{32}{9}\\ &\Rightarrow kt = \ln \tfrac{32}{9}\\ &\Rightarrow t = \frac{1}{k} \ln \tfrac{32}{9}\\ &\Rightarrow t = 8.818\,841\,679 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{t = 8.82 \text{ years (2 dp)}}}.\end{aligned}$$

4. Figure 1 shows a sketch of part of the curve C with equation

$$y = \mathrm{e}^{-2x} + x^2 - 3.$$

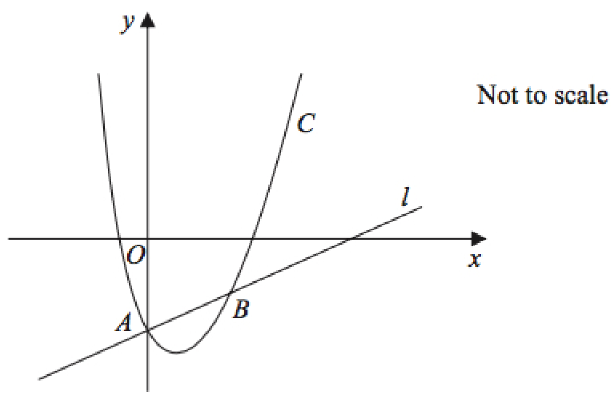

Figure 1: $y = \mathrm{e}^{-2x} + x^2 - 3$

The curve C crosses the y-axis at the point A. The line l is the normal to C at the point A.

(a) Find the equation of l, writing your answer in the form $y = mx + c$, where m and c are constants. (5)

Solution

$$y = \mathrm{e}^{-2x} + x^2 - 3 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -2\mathrm{e}^{-2x} + 2x$$

and, for $x = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -2$ and the gradient of the normal is $\frac{1}{2}$.

Now, $A(0, -2)$ and the the equation is

$$y + 2 = \tfrac{1}{2}(x - 0) \Rightarrow \underline{\underline{y = \tfrac{1}{2}x - 2}}.$$

The line l meets C again at the point B, as shown in Figure 1.

(b) Show that the x-coordinate of B is a solution of (2)

$$x = \sqrt{1 + \tfrac{1}{2}x - \mathrm{e}^{-2x}}.$$

Solution

$$\mathrm{e}^{-2x} + x^2 - 3 = \tfrac{1}{2}x - 2 \Rightarrow x^2 = 1 + \tfrac{1}{2}x - \mathrm{e}^{-2x}$$
$$\Rightarrow \underline{\underline{x = \sqrt{1 + \tfrac{1}{2}x - \mathrm{e}^{-2x}}}},$$

as required.

Using the iterative formula

$$x_{n+1} = \sqrt{1 + \tfrac{1}{2}x_n - \mathrm{e}^{-2x_n}},$$

with $x_1 = 1$,

(c) find x_2 and x_3 to 3 decimal places. (2)

Solution

$x_2 = 1.168\,188\,648$ (FCD) $=$ $\underline{\underline{1.168 \text{ (3 dp)}}}$.

$x_3 = 1.219\,597\,1$ (FCD) $=$ $\underline{\underline{1.220 \text{ (3 dp)}}}$.

5. Figure 2 shows part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2|5 - x| + 3,\ x \geqslant 0.$$

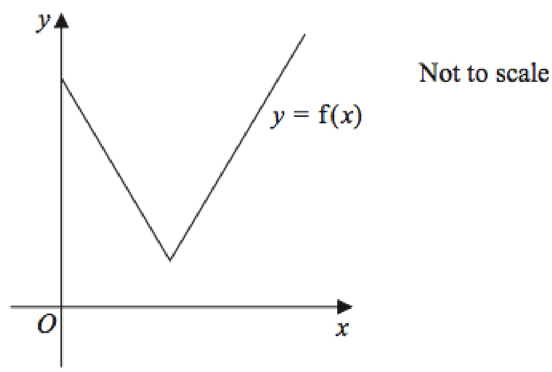

Figure 2: $\mathrm{f}(x) = 2|5 - x| + 3$

Given that the equation $\mathrm{f}(x) = k$, where k is a constant, has exactly one root,

(a) state the set of possible values of k. (2)

Solution

$k = 3$ and $k > 13$.

(b) Solve the equation (4)

$$\mathrm{f}(x) = \tfrac{1}{2}x + 10.$$

Solution

$x < 5$:

$$2(5 - x) + 3 = \tfrac{1}{2}x + 10 \Rightarrow 13 - 2x = \tfrac{1}{2}x + 10$$
$$\Rightarrow \tfrac{5}{2}x = 3$$
$$\Rightarrow x = 1\tfrac{1}{5}.$$

$x \geqslant 5$:

$$-2(5 - x) + 3 = \tfrac{1}{2}x + 10 \Rightarrow -7 + 2x = \tfrac{1}{2}x + 10$$
$$\Rightarrow \tfrac{3}{2}x = 17$$
$$\Rightarrow x = 11\tfrac{1}{3}.$$

The graph with equation

$$y = \mathrm{f}(x)$$

is transformed onto the graph with equation

$$y = 4\,\mathrm{f}(x - 1).$$

The vertex on the graph with equation $y = 4\,\mathrm{f}(x-1)$ has coordinates (p, q).

(c) State the value of p and the value of q. (2)

Solution

$$\begin{aligned} y &= 4\,\mathrm{f}(x - 1) \\ &= 4[2|5 - (x - 1)| + 3] \\ &= 4[2|6 - x| + 3] \\ &= 8|6 - x| + 12; \end{aligned}$$

hence, $\underline{\underline{p = 6}}$ and $\underline{\underline{q = 12}}$.

6. (a) Using the identity for $\tan(A \pm B)$, solve, for $-90^\circ < x < 90^\circ$, (4)

$$\frac{\tan 2x + \tan 32^\circ}{1 - \tan 2x \tan 32^\circ} = 5.$$

Give your answers, in degrees, to 2 decimal places.

Solution

$$\begin{aligned} \frac{\tan 2x + \tan 32^\circ}{1 - \tan 2x \tan 32^\circ} = 5 &\Rightarrow \tan(2x + 32)^\circ = 5 \\ &\Rightarrow 2x + 32 = -101.309\,932\,5,\ 78.690\,067\,53 \text{ (FCD)} \\ &\Rightarrow 2x = -133.309\,932\,5,\ 46.690\,067\,53 \text{ (FCD)} \\ &\Rightarrow x = -65.654\,966\,24,\ 23.345\,033\,76 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = -65.65,\ 23.35 \text{ (2 dp)}}}. \end{aligned}$$

(b) (i) Using the identity for $\tan(A \pm B)$, show that (2)

$$\tan(3\theta - 45^\circ) \equiv \frac{\tan 3\theta - 1}{1 + \tan 3\theta},\ \theta \neq (60n + 45)^\circ,\ n \in \mathbb{Z}.$$

Solution

$$\tan(3\theta - 45^\circ) \equiv \frac{\tan 3\theta - \tan 45^\circ}{\tan 45^\circ + \tan 3\theta}$$
$$\equiv \underline{\underline{\frac{\tan 3\theta - 1}{1 + \tan 3\theta}}},$$

as required.

(ii) Hence solve, for $0^\circ < \theta < 180^\circ$, (5)

$$(1 + \tan 3\theta)\tan(\theta + 28^\circ) = \tan 3\theta - 1.$$

Solution

$$(1 + \tan 3\theta)\tan(\theta + 28^\circ) = \tan 3\theta - 1$$
$$\Rightarrow \quad \tan(\theta + 28^\circ) = \frac{\tan 3\theta - 1}{1 + \tan 3\theta}$$
$$\Rightarrow \quad \tan(\theta + 28^\circ) = \tan(3\theta - 45^\circ)$$
$$\Rightarrow \quad \theta + 28^\circ = 3\theta - 45^\circ + 180n$$
$$\Rightarrow \quad 2\theta = 73^\circ - 180n$$
$$\Rightarrow \quad \theta = 36.5^\circ - 90n;$$

hence, the solutions are $\underline{\underline{36.5^\circ}}$ and $\underline{\underline{126.5^\circ}}$.

7. The curve C has equation

$$y = \frac{\ln(x^2 + 1)}{x^2 + 1},\ x \in \mathbb{R}.$$

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a single fraction, simplifying your answer. (3)

Solution

$$u = \ln(x^2 + 1) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2x}{x^2 + 1}$$
$$v = x^2 + 1 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2x.$$

Finally,

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2+1)\times\frac{2x}{x^2+1} - \ln(x^2+1)\times 2x}{(x^2+1)^2} \\ &= \frac{2x - 2x\ln(x^2+1)}{(x^2+1)^2} \\ &= \underline{\underline{\frac{2x[1-\ln(x^2+1)]}{(x^2+1)^2}}}.\end{aligned}$$

(b) Hence find the exact coordinates of the stationary points of C. (6)

Solution

$$\begin{aligned}\frac{dy}{dx} = 0 &\Rightarrow \frac{2x[1-\ln(x^2+1)]}{(x^2+1)^2} = 0 \\ &\Rightarrow 2x[1-\ln(x^2+1)] = 0 \\ &\Rightarrow x = 0 \text{ or } \ln(x^2+1) = 1 \\ &\Rightarrow x = 0 \text{ or } x^2+1 = \mathrm{e} \\ &\Rightarrow x = 0 \text{ or } x^2 = \mathrm{e}-1 \\ &\Rightarrow x = 0 \text{ or } x = \pm\sqrt{\mathrm{e}-1}.\end{aligned}$$

Finally, the exact coordinates of the stationary points of C are

$$\underline{\underline{(0,0), \left(\sqrt{\mathrm{e}-1}, \frac{1}{\mathrm{e}}\right), \text{ and } \left(-\sqrt{\mathrm{e}-1}, \frac{1}{\mathrm{e}}\right)}}.$$

8. (a) By writing (2)

$$\sec\theta = \frac{1}{\cos\theta},$$

show that

$$\frac{d}{d\theta}(\sec\theta) = \sec\theta\tan\theta.$$

Solution

$$u = 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 0$$

$$v = \cos\theta \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = -\sin\theta.$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}(\sec\theta) &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{\cos\theta}\right) \\
&= \frac{\cos\theta \times 0 - 1 \times (-\sin\theta)}{\cos^2\theta} \\
&= \frac{\sin\theta}{\cos^2\theta} \\
&= \frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta} \\
&= \underline{\underline{\sec\theta\tan\theta}},
\end{aligned}$$

as required.

(b) Given that (5)

$$x = \mathrm{e}^{\sec y},\ x > \mathrm{e},\ 0 < y < \tfrac{1}{2}\pi,$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x\sqrt{\mathrm{g}(x)}},\ x > \mathrm{e},$$

where $\mathrm{g}(x)$ is a function of $\ln x$.

Solution

$$\begin{aligned}
x = \mathrm{e}^{\sec y} &\Rightarrow 1 = \sec y \tan y\, \mathrm{e}^{\sec y}\,\frac{\mathrm{d}y}{\mathrm{d}x} \\
&\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec y \tan y\, \mathrm{e}^{\sec y}} \\
&\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x \ln x \tan y} \\
&\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x \ln x \sqrt{\sec^2 y - 1}} \\
&\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x\sqrt{(\ln x)^2}\sqrt{(\ln x)^2 - 1}} \\
&\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x\sqrt{(\ln x)^4 - (\ln x)^2}}}};
\end{aligned}$$

hence, $\underline{\underline{g(x) = \sqrt{(\ln x)^4 - (\ln x)^2}}}$.

9. *Solutions based entirely on graphical or numerical methods are not acceptable.*

(a) Express (3)

$$\sin\theta - 2\cos\theta$$

in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$.

Give the exact value of R and the value of α, in radians, to 3 decimal places.

Solution

$$R\sin(\theta - \alpha) = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

and so

$$R\cos\alpha = 1 \text{ and } R\sin\alpha = 2.$$

Now,

$$\begin{aligned} R &= \sqrt{(R\sin\theta)^2 + (R\cos\theta)^2} \\ &= \underline{\underline{\sqrt{5}}} \end{aligned}$$

and

$$\begin{aligned} \alpha &= \tan^{-1} 2 \\ &= \underline{\underline{1.107\,148\,718 \text{ (FCD)}}}. \end{aligned}$$

$$M(\theta) = 40 + (3\sin\theta - 6\cos\theta)^2.$$

(b) Find (3)

(i) the maximum value of $M(\theta)$,

Solution

$$\begin{aligned} M(\theta) &= 40 + (3\sin\theta - 6\cos\theta)^2 \\ &= 40 + [3(\sin\theta - 2\cos\theta)]^2 \\ &= 40 + 9(\sin\theta - 2\cos\theta)^2 \\ &= 40 + 9\sin^2(\theta - \alpha) \end{aligned}$$

and the maximum value is

$$40 + 9 \times 5 = \underline{\underline{85}}.$$

(ii) the smallest value of θ, in the range $0 < \theta \leqslant 2\pi$, at which the maximum value of $M(\theta)$ occurs.

Solution
The smallest value is

$$\theta - \alpha = \tfrac{1}{2}\pi \Rightarrow \underline{\underline{\theta = 2.677\,945\,045 \text{ (FCD)}}}.$$

$$N(\theta) = \frac{30}{5 + 2(\sin 2\theta - 2\cos 2\theta)^2}.$$

(c) Find (3)

(i) the maximum value of $N(\theta)$,

Solution
The maximum value of $N(\theta)$ is

$$\frac{30}{5} = \underline{\underline{6}}.$$

(ii) the largest value of θ, in the range $0 < \theta \leqslant 2\pi$, at which the maximum value of $N(\theta)$ occurs.

Solution

$$\begin{aligned} 2\theta - \alpha = \pi + 2\pi &\Rightarrow 2\theta = 3\pi + \alpha \\ &\Rightarrow \theta = \frac{3\pi + \alpha}{2} \\ &\Rightarrow \underline{\underline{\theta = 5.265\,963\,339 \text{ (FCD)}}}. \end{aligned}$$