Dr Oliver Mathematics
Advance Level Further Mathematics
Further Mathematics 1: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1.

$$f(z) \equiv 2z^3 - 4z^2 + 15z - 13.$$

Given that

$$f(z) \equiv (z-1)(2z^2 + az + b),$$

where $a$and b are real constants,

(a) find the value of a and the value of b. (2)

Solution

$\times$	$2z^2$	$+az$	$+b$
z	$2z^3$	$+az^2$	$+bz$
-1	$-2z^2$	$-az$	$-b$

Now,

$$-4 = a - 2 \Rightarrow \underline{\underline{a = -2}}$$

and

$$\underline{\underline{b = 13}}.$$

(b) Hence use algebra to find the three roots of the equation $f(z) = 0$. (4)

Solution

$a = 2$, $b = -2$, and $c = 13$:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{2 \pm \sqrt{-100}}{4} \\ &= \frac{1 \pm 5\text{i}}{2}. \end{aligned}$$

Hence, the three roots of the equation are

$$\underline{\underline{1, \frac{1-5\mathrm{i}}{2}, \text{ and } \frac{1+5\mathrm{i}}{2}}}.$$

2.

$$\mathrm{f}(x) = \tfrac{3}{2}x^2 + \frac{4}{3x} + 2x - 5, \; x < 0.$$

The equation $\mathrm{f}(x) = 0$ has a single root α.

(a) Show that α lies in the interval $[-3, -2.5]$. (2)

Solution

$\mathrm{f}(-3) = 2\frac{1}{18}$
$\mathrm{f}(-2.5) = -1\frac{19}{120}$
Now, $\mathrm{f}(x)$ is continuous (because the question says that $x < 0$) and we change in sign. Therefore, α lies in the interval $[-3, -2.5]$.

(b) Taking -3 as a first approximation to α, apply the Newton-Raphson procedure once to $\mathrm{f}(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places. (5)

Solution

$$\begin{aligned}\mathrm{f}(x) = \tfrac{3}{2}x^2 + \frac{4}{3x} + 2x - 5 &\Rightarrow \mathrm{f}(x) = \tfrac{3}{2}x^2 + \tfrac{4}{3}x^{-1} + 2x - 5 \\ &\Rightarrow \mathrm{f}'(x) = 3x - \tfrac{4}{3}x^{-2} + 2.\end{aligned}$$

Now,

$$\begin{aligned}x_1 &= -3 - \frac{\frac{3}{2}(-3)^2 + \frac{4}{3}(-3)^{-1} + 2(-3) - 5}{3(-3) - \frac{4}{3}(-3)^{-2} + 2} \\ &= -2.712\,435\,233 \text{ (FCD)} \\ &= \underline{\underline{-2.712 \text{ (3 sf)}}}.\end{aligned}$$

(c) Use linear interpolation once on the interval $[-3, -2.5]$ to find another approximation to α, giving your answer to 3 decimal places. (3)

Solution

$$\begin{aligned} x_1 &= \frac{(-3) \times |-1\tfrac{19}{120}| + (-2.5) \times |2\tfrac{1}{18}|}{|2\tfrac{1}{18}| + |-1\tfrac{19}{120}|} \\ &= -2.680\,207\,433 \text{ (FCD)} \\ &= \underline{\underline{-2.680 \text{ (3 sf)}}}. \end{aligned}$$

3. (a) Given that

$$\mathbf{A} = \begin{pmatrix} -2 & 3 \\ 1 & 1 \end{pmatrix} \text{ and } \mathbf{AB} = \begin{pmatrix} -1 & 5 & 12 \\ 3 & -5 & -1 \end{pmatrix},$$

(i) find $\mathbf{A}^{-1}$. (2)

Solution

The determinant is

$$(-2) \times 1 - 3 \times 1 = -5$$

and

$$\mathbf{A}^{-1} = \tfrac{1}{-5} \begin{pmatrix} 1 & -3 \\ -1 & -2 \end{pmatrix} = \underline{\underline{\tfrac{1}{5} \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}}}.$$

(ii) Hence, or otherwise, find the matrix $\mathbf{B}$, giving your answer in its simplest form. (3)

Solution

$$\begin{aligned} \mathbf{B} &= \mathbf{A}^{-1}\mathbf{AB} \\ &= \tfrac{1}{5} \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 5 & 12 \\ 3 & -5 & -1 \end{pmatrix} \\ &= \tfrac{1}{5} \begin{pmatrix} 10 & -20 & -15 \\ 5 & -5 & 10 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 2 & -4 & -3 \\ 1 & -1 & 2 \end{pmatrix}}}. \end{aligned}$$

(b) Given that

$$\mathbf{C} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(i) describe fully the single geometrical transformation represented by the matrix **C**. (2)

> **Solution**
> It is a transformation, 90°, clockwise, about the origin.

(ii) Hence find the matrix $\mathbf{C}^{39}$. (2)

> **Solution**
> $$\mathbf{C}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
> and
> $$\mathbf{C}^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
> Finally,
> $$\begin{aligned} \mathbf{C}^{39} &= (\mathbf{C}^4)^9 \times \mathbf{C} \times \mathbf{C}^2 \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

4. (a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that, for all positive integers n, (4)

$$\sum_{r=1}^{n}(r^2 - r - 8) = \tfrac{1}{3}n(n-a)(n+a),$$

where a is a positive integer to be determined.

> **Solution**

$$
\begin{aligned}
\sum_{r=1}^{n}(r^2 - r - 8) &= \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r - \sum_{r=1}^{n} 8 \\
&= \tfrac{1}{6}n(n+1)(2n+1) - \tfrac{1}{2}n(n+1) - 8n \\
&= \tfrac{1}{6}n[(n+1)(2n+1) - 3(n+1) - 48] \\
&= \tfrac{1}{6}n(2n^2 + 3n + 1 - 3n - 3 - 48) \\
&= \tfrac{1}{6}n(2n^2 - 50) \\
&= \tfrac{1}{3}n(n^2 - 25) \\
&= \underline{\underline{\tfrac{1}{3}n(n-5)(n+5)}};
\end{aligned}
$$

hence, $\underline{\underline{a = 7}}$.

(b) Hence, or otherwise, state the positive value of n that satisfies (1)

$$\sum_{r=1}^{n}(r^2 - r - 8) = 0.$$

Solution

$\underline{\underline{n = 5}}$.

Given that

$$\sum_{r=3}^{17}(kr^3 + r^2 - r - 8) = 6\,710,$$

where k is a constant,

(c) find the exact value of k. (4)

Solution

$$
\begin{aligned}
\sum_{r=3}^{17}(kr^3 + r^2 - r - 8) &= \sum_{r=1}^{17}(kr^3 + r^2 - r - 8) - \sum_{r=1}^{2}(kr^3 + r^2 - r - 8) \\
&= k\sum_{r=1}^{17} r^3 + \sum_{r=1}^{17}(r^2 - r - 8) - \sum_{r=1}^{2}(kr^3 + r^2 - r - 8) \\
&= \tfrac{1}{4}(17)^2(17+1)^2 k + \tfrac{1}{3}(17)(12)(22) - (k - 8 + 8k - 6) \\
&= 23\,409k + 1\,496 - 9k + 14 \\
&= 23\,400k + 1\,510.
\end{aligned}
$$

Now,

$$\sum_{r=3}^{17}(kr^3 + r^2 - r - 8) = 6\,710$$
$$\Rightarrow \quad 23\,400k + 1\,510 = 6\,710$$
$$\Rightarrow \quad 23\,400k = 5\,200$$
$$\Rightarrow \quad \underline{\underline{k = \tfrac{2}{9}}}.$$

5. The rectangular hyperbola H has equation $xy = c^2$, where c is a positive constant. Given that $P\left(ct, \frac{c}{t}\right)$, $t \neq 0$, is a general point on H,

   (a) use calculus to show that the equation of the tangent to H at P can be written as (4)

$$t^2y + x = 2ct.$$

Solution

$$xy = c^2 \Rightarrow y = \frac{c^2}{x}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{x^2},$$

and, at the $P\left(ct, \frac{c}{t}\right)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}.$$

Now,

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct) \Rightarrow t^2y - ct = -x + ct$$
$$\Rightarrow \underline{\underline{t^2y + x = 2ct}}.$$

The points A and B lie on H.

The tangent to H at A and the tangent to H at B meet at the point $\left(-\frac{8c}{5}, \frac{3c}{5}\right)$.

Given that the x-coordinate of A is positive,

   (b) find, in terms of c, the coordinates of A and the coordinates of B. (5)

Solution

Substitute $\left(-\frac{8c}{5}, \frac{3c}{5}\right)$ into the equation $t^2y + x = 2ct$:

$$\begin{aligned} t^2\left(\tfrac{3c}{5}\right) + \left(-\tfrac{8c}{5}\right) = 2ct &\Rightarrow 3t^2 - 8 = 10t \\ &\Rightarrow 3t^2 - 10t - 8 = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -10 \\ \text{multiply to:} & (+3) \times (-8) = -24 \end{array}\right\} -12,\ +2$$

$$\begin{aligned} &\Rightarrow 3t^2 - 12t + 4t - 8 = 0 \\ &\Rightarrow 3t(t-4) + 4(t-4) = 0 \\ &\Rightarrow (3t+2)(t-4) = 0 \\ &\Rightarrow t = -\tfrac{2}{3} \text{ or } t = 4. \end{aligned}$$

Finally,

$$t = -\tfrac{2}{3} \Rightarrow \underline{\underline{B\left(-\frac{2c}{3}, -\frac{3c}{2}\right)}}$$

and

$$t = 4 \Rightarrow \underline{\underline{A\left(4c, \frac{c}{4}\right)}}.$$

6.

$$\mathbf{M} = \begin{pmatrix} 8 & -1 \\ -4 & 2 \end{pmatrix}.$$

(a) Find the value of $\det \mathbf{M}$. (1)

Solution

$$\det \mathbf{M} = (8 \times 2) - [(-1) \times (-4)] = \underline{\underline{12}}.$$

The triangle T has vertices at the points $(4, 1)$, $(6, k)$, and $(12, 1)$, where k is a constant. The triangle T is transformed onto the triangle T' by the transformation represented by the matrix $\mathbf{M}$.
Given that the area of triangle T' is 216 square units,

(b) find the possible values of k. (5)

Solution

$$\begin{aligned}216 &= 12 \times \tfrac{1}{2} \times |k-1| \times 8\\ &\Rightarrow |k-1| = 4.5\\ &\Rightarrow k-1 = -4.5 \text{ or } k-1 = 4.5\\ &\Rightarrow \underline{\underline{k = -3.5 \text{ or } k = 5.5}}.\end{aligned}$$

7. The parabola C has equation $y^2 = 4ax$, where a is a positive constant.
The point S is the focus of C.
The straight line l passes through the point S and meets the directrix of C at the point D.
Given that the y-coordinate of D is $\dfrac{24a}{5}$,

(a) show that an equation of the line l is (2)

$$12x + 5y = 12a.$$

Solution

Now, $S(a, 0)$ and $D(-a, \frac{24a}{5})$. The gradient of l is

$$\frac{\frac{24a}{5} - 0}{-a - a} = -\tfrac{12}{5}$$

and an equation of the line l is

$$\begin{aligned}y - 0 = -\tfrac{12}{5}(x - a) &\Rightarrow 5y = -12(x-a)\\ &\Rightarrow 5y = -12x + 12a\\ &\Rightarrow \underline{\underline{12x + 5y = 12a}},\end{aligned}$$

as required.

The point $P(ak^2, 2ak)$, where k is a positive constant, lies on the parabola C.
Given that the line segment SP is perpendicular to l,

(b) find, in terms of a, the coordinates of the point P. (6)

Solution

The gradient of PS is

$$\frac{2ak - 0}{ak^2 - a} = \frac{2k}{k^2 - 1}$$

and the equation of the line perpendicular to l is $\frac{5}{12}$. Now, the two expressions are equal:

$$\begin{aligned}\frac{2k}{k^2-1} = \frac{5}{12} &\Rightarrow 24k = 5(k^2 - 1)\\ &\Rightarrow 24k = 5k^2 - 5\\ &\Rightarrow 5k^2 - 24k - 5 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -24\\ \text{multiply to:} & (+5) \times (-5) = -25\end{array}\right\} -25,\ +1$$

$$\begin{aligned}&\Rightarrow 5k^2 - 25k + k - 5 = 0\\ &\Rightarrow 5k(k-5) + 1(k-5) = 0\\ &\Rightarrow (5k+1)(k-5) = 0\\ &\Rightarrow k = -\tfrac{1}{5} \text{ or } k = 5.\end{aligned}$$

Finally, $k > 0$ and $\underline{\underline{P(25a, 10a)}}$.

8. Prove by induction that (6)

$$f(n) = 2^{n+2} + 3^{2n+1}$$

is divisible by 7 for all positive integers n.

Solution

$\underline{n = 1}$:

$$f(1) = 2^3 + 3^3 = 8 + 27 = 35 = 7 \times 5$$

and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$f(k) = 2^{k+2} + 3^{2k+1}$$

is divisible by 7.

$$\begin{aligned}
\mathrm{f}(k+1)-\mathrm{f}(k) &= \left[2^{(k+1)+2}+3^{2(k+1)+1}\right]-\left[2^{k+2}+3^{2k+1}\right] \\
&= \left[2^{k+3}+3^{2k+3}\right]-\left[2^{k+2}+3^{2k+1}\right] \\
&= \left(2^{k+3}-2^{k+2}\right)+\left(3^{2k+3}-3^{2k+1}\right) \\
&= 2^{k+2}(2-1)+3^{2k+1}(3^2-1) \\
&= 2^{k+2}+8\times 3^{2k+1} \\
&= \left(2^{k+2}+3^{2k+1}\right)+7\times 3^{2k+1},
\end{aligned}$$

and so

$$\mathrm{f}(k+1)=2\,\mathrm{f}(k)+7\times 3^{2k+1},$$

which is divisible by 7.

Hence, by mathematical induction, the expression is true for all positive integers n, as required.

9. (a) Given that

$$\frac{3w+7}{5}=\frac{p-4\mathrm{i}}{3-\mathrm{i}},$$

where p is a real constant

(i) express w in the form $a+b\mathrm{i}$, where a and b are real constants. Give your answer in its simplest form in terms of p. (5)

Solution

$$\begin{aligned}
\frac{3w+7}{5}=\frac{p-4\mathrm{i}}{3-\mathrm{i}} &\Rightarrow \frac{3w+7}{5}=\frac{p-4\mathrm{i}}{3-\mathrm{i}}\times\frac{3+\mathrm{i}}{3+\mathrm{i}} \\
&\Rightarrow \frac{2(3w+7)}{10}=\frac{(p+4)+(3p-12)\mathrm{i}}{10} \\
&\Rightarrow 2(3w+7)=(3p+4)+(p-12)\mathrm{i} \\
&\Rightarrow 6w+14=(3p+4)+(p-12)\mathrm{i} \\
&\Rightarrow 6w=(3p-10)+(p-12)\mathrm{i} \\
&\Rightarrow w=\underline{\underline{\frac{3p-10}{6}+\left(\frac{p-12}{6}\right)\mathrm{i}}}.
\end{aligned}$$

Given that $\arg w=-\frac{\pi}{2}$,

(ii) find the value of p. (1)

Solution

$$\frac{3p-10}{6} = 0 \Rightarrow \underline{\underline{p = \tfrac{10}{3}}}.$$

(b) Given that (6)

$$(z + 1 - 2\mathrm{i})^* = 4\mathrm{i}z,$$

find z, giving your answer in the form $z = x + \mathrm{i}y$, where x and y are real constants.

Solution

$$\begin{aligned}(z + 1 - 2\mathrm{i})^* = 4\mathrm{i}z &\Rightarrow (x + \mathrm{i}y + 1 - 2\mathrm{i})^* = 4\mathrm{i}(x + \mathrm{i}y)\\ &\Rightarrow [(x+1) + (y-2)\mathrm{i}]^* = -4y + 4x\mathrm{i}\\ &\Rightarrow x + 1 - (y-2)\mathrm{i} = -4y + 4x\mathrm{i};\end{aligned}$$

so,

$$\begin{aligned}x + 1 = -4y &\Rightarrow x + 4y = -1 \quad (1)\\ -y + 2 = 4x &\Rightarrow 4x + y = 2. \quad (2)\end{aligned}$$

(2) $-\, 4 \times$ (1):

$$\begin{aligned}-15y = 6 &\Rightarrow y = -\tfrac{2}{5}\\ &\Rightarrow x + 1 = \tfrac{8}{5}\\ &\Rightarrow x = \tfrac{3}{5};\end{aligned}$$

hence,

$$\underline{\underline{z = \tfrac{3}{5} - \tfrac{2}{5}\mathrm{i}}}.$$