Dr Oliver Mathematics
Chain Rule: Part 1

1. If

$$\begin{aligned} y &= \tan u, \\ u &= v - \frac{1}{v}, \text{ and} \\ v &= \ln x, \end{aligned}$$

what is the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at $x = \mathrm{e}$?

Solution

$$\begin{aligned} y = \tan u &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}u} = \sec^2 u, \\ u = v - v^{-1} &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}v} = 1 + v^{-2}, \text{ and} \\ v = \ln x &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x}. \end{aligned}$$

Now,

$$\begin{aligned} x = \mathrm{e} &\Rightarrow v = \ln \mathrm{e} = 1 \\ &\Rightarrow u = 1 - \frac{1}{1} = 0, \text{ and} \\ &\Rightarrow y = \tan 0 = 0. \end{aligned}$$

Finally,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}v} \cdot \frac{\mathrm{d}v}{\mathrm{d}x} \\ &= (\sec^2 u) \cdot \left(1 + v^{-2}\right) \cdot \left(\frac{1}{x}\right) \\ &= (\sec^2 0) \cdot \left(1 + 1^{-2}\right) \cdot \left(\frac{1}{\mathrm{e}}\right) \\ &= (1) \cdot (2) \cdot \left(\frac{1}{\mathrm{e}}\right) \\ &= \underline{\underline{\frac{2}{\mathrm{e}}}}. \end{aligned}$$