Dr Oliver Mathematics
Advance Level Further Mathematics
Core Pure Mathematics 1: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1.
$$f(z) = z^4 + az^3 + bz^2 + cz + d,$$

where a, b, c, and d are real constants.

Given that

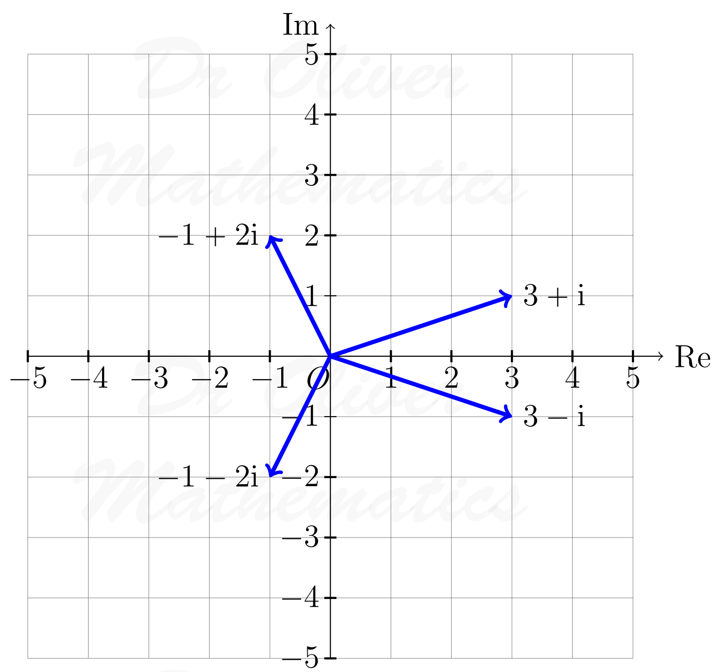
$$-1 + 2\text{i} \text{ and } 3 - \text{i}$$

are two roots of the equation $f(z) = 0$,

(a) show all the roots of $f(z) = 0$ on a single Argand diagram, (4)

Solution

The solutions are $-1 \pm 2\text{i}$ and $3 \pm \text{i}$.



(b) find the values of a, b, c, and d. (5)

Solution

$\times$	z	$+1$	$-2\mathrm{i}$
z	z^2	$+z$	$-2\mathrm{i}z$
$+1$	$+z$	$+1$	$-2\mathrm{i}$
$+2\mathrm{i}$	$+2\mathrm{i}z$	$+2\mathrm{i}$	$+4$

So

$$[z-(-1+2\mathrm{i})][z-(-1-2\mathrm{i})] = z^2+2z+5.$$

$\times$	z	-3	$-\mathrm{i}$
z	z^2	$-3z$	$-\mathrm{i}z$
-3	$-3z$	$+9$	$+3\mathrm{i}$
$+\mathrm{i}$	$+\mathrm{i}z$	$-3\mathrm{i}$	$+1$

So

$$[z-(3+\mathrm{i})][z-(3-\mathrm{i})] = z^2-6z+10.$$

$\times$	z^2	$+2z$	$+5$
z^2	z^4	$+2z^2$	$+5z$
$-6z$	$-6z^3$	$-12z^2$	$-30z$
$+10$	$+10z^2$	$+20z$	$+50$

so

$$(z^2+2z+5)(z^2-6z+10) = \underline{\underline{z^4-4z^3+3z^2-10z+50}},$$

hence, $\underline{\underline{a=-4}}$, $\underline{\underline{b=3}}$, $\underline{\underline{c=-10}}$, and $\underline{\underline{d=50}}$.

2. Show that (7)

$$\int_0^\infty \frac{8x-12}{(2x^2+3)(x+1)}\,\mathrm{d}x = \ln k,$$

where k is a rational number to be found.

Solution

$$\frac{8x-12}{(2x^2+3)(x+1)} \equiv \frac{Ax+B}{2x^2+3}+\frac{C}{x+1}$$
$$\equiv \frac{(Ax+B)(x+1)+C(2x^2+3)}{(2x^2+3)(x+1)}$$

and hence

$$8x-12 \equiv (Ax+B)(x+1)+C(2x^2+3).$$

$\underline{x=-1}$: $-20 = 5C \Rightarrow C = -4$.
$\underline{x=0}$: $-12 = B-12 \Rightarrow B = 0$.
$\underline{x=1}$: $-4 = 2A-20 \Rightarrow A = 8$.
Hence,

$$\frac{8x-12}{(2x^2+3)(x+1)} \equiv \frac{8x}{2x^2+3}-\frac{4}{x+1}.$$

Now,

$$\begin{aligned}
\int_0^k \frac{8x-12}{(2x^2+3)(x+1)}\,\mathrm{d}x &= \int_0^k \left[\frac{8x}{2x^2+3}-\frac{4}{x+1}\right]\mathrm{d}x \\
&= \big[2\ln|2x^2+3|-4\ln|x+1|\big]_{x=0}^{k} \\
&= \big(2\ln|2k^2+3|-4\ln|k+1|\big)-(2\ln 3-4\ln 1) \\
&= \ln\left|\frac{(2k^2+3)^2}{(k+1)^4}\right|-\ln 9 \\
&= \ln\left|\frac{4k^4+12k^2+9}{k^4+4k^3+6k^2+4k+1}\right|-\ln 9 \\
&= \ln\left|\frac{4+\frac{12}{k^2}+\frac{9}{k^4}}{1+\frac{4}{k}+\frac{6}{k^2}+\frac{4}{k^3}+\frac{1}{k^4}}\right|-\ln 9
\end{aligned}$$

and so

$$\begin{aligned}
\int_0^\infty \frac{8x-12}{(2x^2+3)(x+1)}\,\mathrm{d}x &= \lim_{k\to\infty}\left[\int_0^k \left[\frac{8x}{2x^2+3}-\frac{4}{x+1}\right]\mathrm{d}x\right] \\
&= \lim_{k\to\infty}\left[\ln\left|\frac{4+\frac{12}{k^2}+\frac{9}{k^4}}{1+\frac{4}{k}+\frac{6}{k^2}+\frac{4}{k^3}+\frac{1}{k^4}}\right|-\ln 9\right] \\
&= \ln 4-\ln 9 \\
&= \underline{\underline{\ln \tfrac{4}{9}}}.
\end{aligned}$$

3. Figure 1 shows the design for a table top in the shape of a rectangle $ABCD$. The length of the table, AB, is 1.2 m.

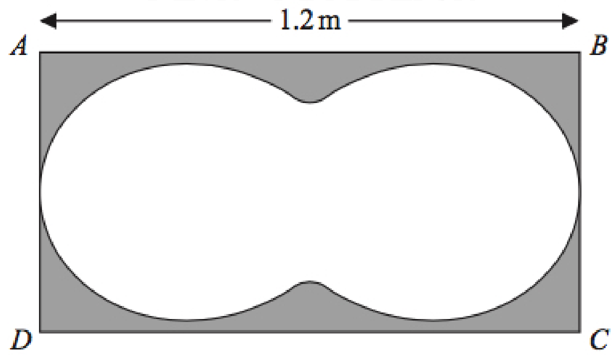


Figure 1: a table top

The area inside the closed curve is made of glass and the surrounding area, shown shaded in Figure 1, is made of wood.

The perimeter of the glass is modelled by the curve with polar equation

$$r = 0.4 + a\cos 2\theta,\ 0 \leqslant \theta < 2\pi,$$

where a is a constant.

(a) Show that $a = 0.2$. (2)

Solution

The glass is, horizontally, 1.2 m across:

$$\begin{aligned} 1.2 = 2(0.4 + a\cos 0) &\Rightarrow 0.4 + a = 0.6 \\ &\Rightarrow \underline{\underline{a = 0.2}}, \end{aligned}$$

as required.

Hence, given that $AD = 60$ cm,

(b) find that area of the wooden part of the table top, giving your answer in m^2 to 3 significant figures. (8)

Solution

$$
\begin{aligned}
\text{Wooden part} &= \text{whole table top} - \text{glass part} \\
&= (1.2 \times 0.6) - \tfrac{1}{2}\pi \int_0^{2\pi} r^2 \, \mathrm{d}\theta \\
&= 0.72 - \pi \int_0^{\pi} (0.4 + 0.2\cos 2\theta)^2 \, \mathrm{d}\theta \\
&= 0.72 - \pi \int_0^{\pi} (0.16 + 0.16\cos 2\theta + 0.04\cos^2 2\theta) \, \mathrm{d}\theta \\
&= 0.72 - \pi \int_0^{\pi} (0.16 + 0.16\cos 2\theta + 0.04 \cdot \tfrac{1}{2}(\cos 4\theta + 1)) \, \mathrm{d}\theta \\
&= 0.72 - \pi \int_0^{\pi} (0.18 + 0.16\cos 2\theta + 0.02\cos 4\theta) \, \mathrm{d}\theta \\
&= 0.72 - \pi \left[0.18\theta + 0.08\sin 2\theta + 0.005\sin 4\theta\right]_{\theta=0}^{\pi} \\
&= 0.72 - \pi \left\{(0.18 + 0 + 0) - (0 + 0 + 0)\right\} \\
&= 0.72 - 0.18\pi \\
&= 0.154\,513\,322\,4 \text{ (FCD)} \\
&= \underline{\underline{0.155 \text{ m}^2}}.
\end{aligned}
$$

4. Prove that, for $n \in \mathbb{Z}$, $n \geqslant 0$, (5)

$$
\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{(n+a)(n+b)}{c(n+2)(n+3)},
$$

where a, b, and c are integers to be found.

Solution

$$
\begin{aligned}
\frac{1}{(r+1)(r+2)(r+3)} &\equiv \frac{A}{r+1} + \frac{B}{r+2} + \frac{C}{r+3} \\
&\equiv \frac{A(r+2)(r+3) + B(r+1)(r+3) + C(r+1)(r+2)}{(r+1)(r+2)(r+3)}
\end{aligned}
$$

and hence

$$
1 = A(r+2)(r+3) + B(r+1)(r+3) + C(r+1)(r+2).
$$

$\underline{r = -1}$: $1 = -2A \Rightarrow A = \frac{1}{2}$.
$\underline{r = -2}$: $1 = -B \Rightarrow B = -1$.

$\underline{r=-3}$: $1 = -2C \Rightarrow C = \frac{1}{2}$.
Hence,

$$\frac{1}{(r+1)(r+2)(r+3)} \equiv \frac{\frac{1}{2}}{r+1} - \frac{1}{r+2} + \frac{\frac{1}{2}}{r+3}.$$

Now,

$$\begin{aligned}
&\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} \\
&= \sum_{r=0}^{n} \left[\frac{\frac{1}{2}}{r+1} - \frac{1}{r+2} + \frac{\frac{1}{2}}{r+3} \right] \\
&= \left(\frac{\frac{1}{2}}{1} - \frac{1}{2} + \frac{\frac{1}{2}}{3} \right) + \left(\frac{\frac{1}{2}}{2} - \frac{1}{3} + \frac{\frac{1}{2}}{4} \right) + \left(\frac{\frac{1}{2}}{3} - \frac{1}{4} + \frac{\frac{1}{2}}{5} \right) \\
&\quad + \ldots + \left(\frac{\frac{1}{2}}{n} - \frac{1}{n+1} + \frac{\frac{1}{2}}{n+2} \right) + + \left(\frac{\frac{1}{2}}{n+1} - \frac{1}{n+2} + \frac{\frac{1}{2}}{n+3} \right) \\
&= \underbrace{\left(\frac{\frac{1}{2}}{1} - \frac{1}{2} \right)}_{\text{from the 1st term}} + \underbrace{\left(\frac{\frac{1}{2}}{2} \right)}_{\text{from the 2nd term}} + \underbrace{\left(\frac{\frac{1}{2}}{n+2} \right)}_{\text{from the } (n-1)\text{th term}} + \underbrace{\left(-\frac{1}{n+2} + \frac{\frac{1}{2}}{n+3} \right)}_{\text{from the } n\text{th term}} \\
&= \frac{1}{4} - \frac{1}{2(n+2)} + \frac{1}{2(n+3)} \\
&= \frac{(n+2)(n+3) - 2(n+3) + 2(n+2)}{4(n+2)(n+3)} \\
&= \frac{(n^2+5n+6) - 2n - 6 + 2n + 4}{4(n+2)(n+3)} \\
&= \frac{n^2+5n+4}{4(n+2)(n+3)} \\
&= \underline{\underline{\frac{(n+1)(n+4)}{4(n+2)(n+3)}}};
\end{aligned}$$

so, $\underline{\underline{a=1}}$, $\underline{\underline{b=4}}$, and $\underline{\underline{c=4}}$.

5. A tank at a chemical plant has a capacity of 250 litres. The tank initially contains 100 litres of pure water.

   Salt water enters the tank at a rate of 3 litres every minute. Each litre of salt water entering the tank contains 1 gram of salt.

   It is assumed that the salt water mixes instantly with the contents of the tank upon entry.

At the instant when the salt water begins to enter the tank, a value is opened at the bottom the of the tank and the solution in the tank flows out a rate of 2 litres every minute.

Given that there are S grams of salt in the tank after t minutes,

(a) show that the situation can be modelled by the differential equation (4)

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 3 - \frac{2S}{100+t}.$$

Solution

First, salt enters the tank at a rate of $3 \times 1 = 3$ g/minute.

Second, 3 litres are entering the tank and 2 litres are leaving the tank; hence, 1 litre is entering the tank net which and so $(100 + t)$ litres after t minutes.

Third, 2 litres are leaving the tank and, if the salt in the tank is S grams, the concentration is

$$\frac{S}{100+t} \text{ g/l.}$$

Hence, salt leaves the tank at a rate of

$$\frac{2S}{100+t} \text{ g/minute.}$$

Putting that all together,

$$\underline{\underline{\frac{\mathrm{d}S}{\mathrm{d}t} = 3 - \frac{2S}{100+t}}}.$$

(b) Hence find the number of grams of salt in the tank after 10 minutes. (5)

Solution

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 3 - \frac{2S}{100+t} \Rightarrow \frac{\mathrm{d}S}{\mathrm{d}t} + \frac{2S}{100+t} = 3$$

$$\begin{aligned}\text{Integrating Factor (IF)} &= e^{\int \frac{2}{100+t}\,dt} \\ &= e^{2\ln(100+t)} \\ &= e^{\ln(100+t)^2} \\ &= (100+t)^2\end{aligned}$$

$$\begin{aligned}&\Rightarrow (100+t)^2\frac{dS}{dt} + 2(100+t)S = 3(100+t)^2 \\ &\Rightarrow \frac{d}{dt}\left[(100+t)^2 S\right] = 3(100+t)^2 \\ &\Rightarrow (100+t)^2 S = \int 3(100+t)^2\,dt \\ &\Rightarrow (100+t)^2 S = (100+t)^3 + c.\end{aligned}$$

Now,

$$\begin{aligned}t = 0, S = 0 &\Rightarrow 0 = 100^3 + c \\ &\Rightarrow c = -1\,000\,000\end{aligned}$$

and

$$(100+t)^2 S = (100+t)^3 - 1\,000\,000.$$

Finally,

$$\begin{aligned}t = 10 &\Rightarrow (110)^2 S = (110)^3 - 1\,000\,000 \\ &\Rightarrow 12\,100\,S = 1\,331\,000 - 1\,000\,000 \\ &\Rightarrow 12\,100\,S = 331\,000 \\ &\Rightarrow \underline{\underline{S = 27\tfrac{43}{121}\text{ g}}}.\end{aligned}$$

When the concentration of the salt in the tank reaches 0.9 grams per litre, the value at the bottom of the tank must be closed.

(c) Find, to the nearest minute, when the valve would need to be closed. (3)

Solution

Now,

$$(100+t)^2 S = (100+t)^3 - 1\,000\,000 \Rightarrow S = (100+t) - \frac{1\,000\,000}{(100+t)^2}$$

and

$$\frac{(100+t) - \frac{1\,000\,000}{(100+t)^2}}{100+t} = 0.9 \Rightarrow (100+t) - \frac{1\,000\,000}{(100+t)^2} = 0.9(100+t)$$

$$\Rightarrow \frac{(100+t)^3 - 1\,000\,000}{(100+t)^2} = 0.9(100+t)$$

$$\Rightarrow 0.1(100+t)^3 = 1\,000\,000$$

$$\Rightarrow (100+t)^3 = 10\,000\,000$$

$$\Rightarrow 100 + t = 215.443\,469 \text{ (FCD)}$$

$$\Rightarrow t = 115.443\,469 \text{ (FCD)}$$

$$\Rightarrow \underline{\underline{t = 115 \text{ minutes (3 sf)}}}.$$

(d) Evaluate the model (1)

Solution

E.g., *It is assumed that the salt water mixes instantly with the contents of the tank upon entry*: unlikely that it is instantaneous.

6. Prove by induction that for all positive integers n (6)

$$\mathrm{f}(n) = 3^{2n+4} - 2^{2n}$$

is divisible by 5.

Solution

$\underline{n = 1}$:

$$\begin{aligned} \mathrm{f}(1) &= 3^6 - 2^2 \\ &= 729 - 4 \\ &= 725 \\ &= 5 \times 145, \end{aligned}$$

and so we can see $n = 1$ is true.

Suppose $n = k$ is true, i.e.,

$$\mathrm{f}(k) = 3^{2k+4} - 2^{2k}$$

is divisible by 5. Now,

$$\begin{aligned}
\mathrm{f}(k+1) &= 3^{2(k+1)+4} - 2^{2(k+1)} \\
&= 3^{2k+6} - 2^{2k+2} \\
&= 3^2 \cdot 3^{2k+4} - 2^2 \cdot 2^{2k} \\
&= 9 \cdot 3^{2k+4} - 4 \cdot 2^{2k} \\
&= \underbrace{5}_{\div 5} \cdot 3^{2k+4} + 4(\underbrace{3^{2k+4} - 2^{2k}}_{\div 5}),
\end{aligned}$$

and so $\mathrm{f}(k+1)$ is divisible by 5.

Hence, by mathematical induction, $\mathrm{f}(n) = 3^{2n+4} - 2^{2n}$ is divisible by 5 for all $n \in \mathbb{N}$.

7. The line l_1 has equation

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-4}{3}.$$

The line l_2 has equation

$$\mathbf{r} = \mathbf{i} + 3\mathbf{k} + t(\mathbf{i} - \mathbf{j} + 2\mathbf{k}),$$

where t is a scalar parameter.

(a) Show that l_1 and l_2 lie in the same plane. (3)

Solution

The line l_1 has equation

$$\begin{aligned}
\mathbf{r} &= \mathbf{i} - \mathbf{j} + 4\mathbf{k} + s(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \\
&= (1+2s)\mathbf{i} + (-1-s)\mathbf{j} + (4+3s)\mathbf{k}
\end{aligned}$$

and we equate the x, y, and z terms:

$$\begin{aligned}
1 + 2s &= 1 + t \quad (1) \\
-1 - s &= -t \quad (2) \\
4 + 3s &= 3 + 2t \quad (3)
\end{aligned}$$

Now, (1) + (2):

$$\begin{aligned}
s = 1 &\Rightarrow 3 = 1 + t \\
&\Rightarrow t = 2.
\end{aligned}$$

Check them in equation (3):

$$4 + 3s = 7 \text{ and } 3 + 2t = 7 \checkmark.$$

So, l_1 and l_2 lie in the same plane.

(b) Write down a vector equation for the plane containing l_1 and l_2. (1)

Solution

E.g.,

$$\underline{\underline{\mathbf{r} = \mathbf{i} - \mathbf{j} + 4\mathbf{k} + s(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} - \mathbf{j} + 2\mathbf{k})}}.$$

(c) Find, to the nearest degree, the acute angle between l_1 and l_2. (3)

Solution

$$\begin{aligned} & \mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \\ \Rightarrow\quad & 2 + 1 + 6 = \sqrt{2^2 + (-1)^2 + 3^2} \cdot \sqrt{1^2 + (-1)^2 + 2^2} \cdot \cos\theta \\ \Rightarrow\quad & 9 = \sqrt{14} \cdot \sqrt{6} \cdot \cos\theta \\ \Rightarrow\quad & \cos\theta = \frac{9}{\sqrt{14} \cdot \sqrt{6}} \\ \Rightarrow\quad & \theta = 10.893\,394\,65 \text{ (FCD)} \\ \Rightarrow\quad & \underline{\underline{\theta = 11^\circ \text{ (nearest degree)}}}. \end{aligned}$$

8. A scientist is studying the effect of introducing a population of white-clawed crayfish into a population of signal crayfish.

At time t years, the number of white-clawed crayfish, w, and the signal crayfish, s, are modelled by the differential equation

$$\begin{aligned} \frac{\mathrm{d}w}{\mathrm{d}t} &= \tfrac{5}{2}(w - s) \\ \frac{\mathrm{d}s}{\mathrm{d}t} &= \tfrac{2}{5}w - 90\mathrm{e}^{-t}. \end{aligned}$$

(a) Show that (3)

$$2\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} - 5\frac{\mathrm{d}w}{\mathrm{d}t} + 2w = 450\mathrm{e}^{-t}.$$

Solution

$$\begin{aligned}\frac{\mathrm{d}^2w}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{5}{2}(w-s)\right] \\ &= \tfrac{5}{2}\frac{\mathrm{d}w}{\mathrm{d}t} - \tfrac{5}{2}\frac{\mathrm{d}s}{\mathrm{d}t} \\ &= \tfrac{5}{2}\frac{\mathrm{d}w}{\mathrm{d}t} - \tfrac{5}{2}(\tfrac{2}{5}w - 90\mathrm{e}^{-t}) \\ &= \tfrac{5}{2}\frac{\mathrm{d}w}{\mathrm{d}t} - w + 225\mathrm{e}^{-t}.\end{aligned}$$

Now,

$$\begin{aligned} & 2\frac{\mathrm{d}^2w}{\mathrm{d}t^2} - 5\frac{\mathrm{d}w}{\mathrm{d}t} + 2w \\ = \; & 2\left[\tfrac{5}{2}\frac{\mathrm{d}w}{\mathrm{d}t} - w + 225\mathrm{e}^{-t}\right] - 5\left[\tfrac{5}{2}(w-s)\right] + 2w \\ = \; & \left[2\frac{\mathrm{d}w}{\mathrm{d}t} - 2w + 450\mathrm{e}^{-t}\right] - \left[\tfrac{25}{2}(w-s)\right] + 2w \\ = \; & \underline{\underline{450\mathrm{e}^{-t}}},\end{aligned}$$

as required.

(b) Find a general solution for the number of white-clawed crayfish at time t years. (6)

Solution

Complementary function:

$$2m^2 - 5m + 2 = 0 \Rightarrow (2m-1)(m-2) = 0 \Rightarrow m = \tfrac{1}{2} \text{ or } m = 2$$

and hence the complementary function is

$$w = A\mathrm{e}^{\frac{1}{2}t} + B\mathrm{e}^{2t}.$$

Particular integral: try

$$w = C\mathrm{e}^{-t} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -C\mathrm{e}^{-t} \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = C\mathrm{e}^{-t}.$$

Substitute into the differential equation:

$$2C + 5C + 2C = 450 \Rightarrow C = 50.$$

Hence the particular integral is $w = 50\mathrm{e}^{-t}$.

General solution: hence the general solution is

$$\underline{\underline{w = A\mathrm{e}^{\frac{1}{2}t} + B\mathrm{e}^{2t} + 50\mathrm{e}^{-t}}}.$$

(c) Find a general solution for the number of signal crayfish at time t years. (2)

Solution

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t} = \tfrac{5}{2}(w - s) \Rightarrow \tfrac{2}{5}\frac{\mathrm{d}w}{\mathrm{d}t} &= w - s \\ \Rightarrow s &= w - \tfrac{2}{5}\frac{\mathrm{d}w}{\mathrm{d}t} \\ \Rightarrow s &= (A\mathrm{e}^{\frac{1}{2}t} + B\mathrm{e}^{2t} + 50\mathrm{e}^{-t}) - \tfrac{2}{5}\frac{\mathrm{d}w}{\mathrm{d}t}.\end{aligned}$$

Now,

$$\begin{aligned} w &= A\mathrm{e}^{\frac{1}{2}t} + B\mathrm{e}^{2t} + 50\mathrm{e}^{-t} \\ \Rightarrow \quad \frac{\mathrm{d}w}{\mathrm{d}t} &= \tfrac{1}{2}A\mathrm{e}^{\frac{1}{2}t} + 2B\mathrm{e}^{2t} - 50\mathrm{e}^{-t}\end{aligned}$$

which means

$$\begin{aligned} s &= (A\mathrm{e}^{\frac{1}{2}t} + B\mathrm{e}^{2t} + 50\mathrm{e}^{-t}) - \tfrac{2}{5}\left[\tfrac{1}{2}A\mathrm{e}^{\frac{1}{2}t} + 2B\mathrm{e}^{2t} - 50\mathrm{e}^{-t}\right] \\ &= (A\mathrm{e}^{\frac{1}{2}t} + B\mathrm{e}^{2t} + 50\mathrm{e}^{-t}) - (\tfrac{1}{5}A\mathrm{e}^{\frac{1}{2}t} + \tfrac{4}{5}B\mathrm{e}^{2t} - 20\mathrm{e}^{-t}) \\ &= \underline{\underline{\tfrac{4}{5}A\mathrm{e}^{\frac{1}{2}t} + \tfrac{1}{5}B\mathrm{e}^{2t} + 70\mathrm{e}^{-t}}}.\end{aligned}$$

The model predicts that, at time T years, the population of white-clawed crayfish will have died out.

Given that $w = 65$ and $s = 85$ when $t = 0$,

(d) find the value of T, giving your answer to 3 decimal places. (6)

Solution

Well,

$$\begin{aligned} w = 65, t = 0 \Rightarrow 65 &= A + B + 50 \\ \Rightarrow A + B &= 15 \quad (1) \end{aligned}$$

and

$$\begin{aligned} s = 85, t = 0 \Rightarrow 85 &= \tfrac{4}{5}A + \tfrac{1}{5}B + 70 \\ \Rightarrow \tfrac{4}{5}A + \tfrac{1}{5}B &= 15 \\ \Rightarrow 4A + B &= 75 \quad (2). \end{aligned}$$

Do (2) − (1):

$$\begin{aligned} 3A = 60 \Rightarrow A &= 20 \\ \Rightarrow B &= -5. \end{aligned}$$

So,

$$w = 20\mathrm{e}^{\frac{1}{2}t} - 5\mathrm{e}^{2t} + 50\mathrm{e}^{-t}.$$

Finally,

$$\begin{aligned} w = 0 &\Rightarrow 20\mathrm{e}^{\frac{1}{2}t} - 5\mathrm{e}^{2t} + 50\mathrm{e}^{-t} = 0 \\ &\Rightarrow 20\mathrm{e}^{\frac{3}{2}t} - 5\mathrm{e}^{3t} + 50 = 0 \\ &\Rightarrow 5\mathrm{e}^{3t} - 20\mathrm{e}^{\frac{3}{2}t} - 50 = 0 \\ &\Rightarrow \mathrm{e}^{3t} - 4\mathrm{e}^{\frac{3}{2}t} - 10 = 0 \\ &\Rightarrow \mathrm{e}^{3t} - 4\mathrm{e}^{\frac{3}{2}t} + 4 = 14 \\ &\Rightarrow (\mathrm{e}^{\frac{3}{2}t} - 2)^2 = 14 \\ &\Rightarrow \mathrm{e}^{\frac{3}{2}t} - 2 = \sqrt{14} \text{ (why can we do without the minus sign?)} \\ &\Rightarrow \mathrm{e}^{\frac{3}{2}t} = 2 + \sqrt{14} \\ &\Rightarrow \tfrac{3}{2}t = \ln(2 + \sqrt{14}) \\ &\Rightarrow t = \tfrac{2}{3}\ln(2 + \sqrt{14}) \\ &\Rightarrow t = 1.165\,165\,275 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{t = 1.17 \text{ (3 sf)}}}. \end{aligned}$$

(e) Suggest a limitation of the model. (1)

Solution

E.g., When the white-clawed crayfish have died out, the model will not be valid.