Dr Oliver Mathematics
Further Mathematics
Moments Part 2
Past Examination Questions

This booklet consists of 27 questions across a variety of examination topics. The total number of marks available is 261.

1. A uniform ladder AB, of mass m and length $2a$, has one end A on rough horizontal ground. The other end B rests against a smooth vertical wall. The ladder is in a vertical plane perpendicular to the wall. The ladder makes an angle α with the horizontal, where $\tan \alpha = \frac{4}{3}$. A child of mass $2m$ stands on the ladder at C where $AC = \frac{1}{2}a$, as shown in Figure 1. (9)

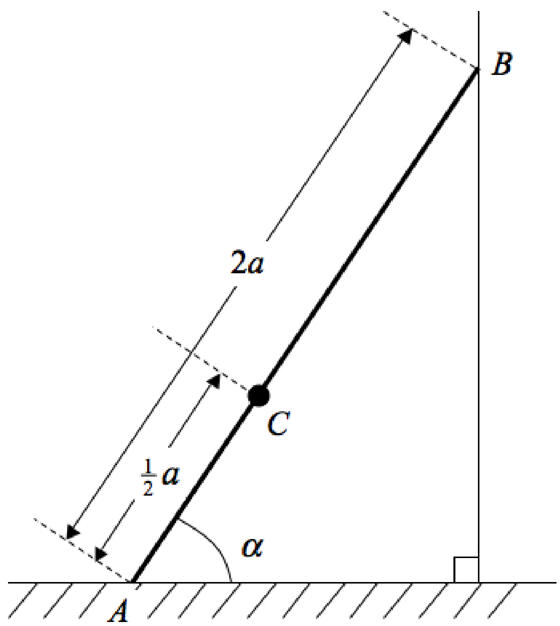


Figure 1: a uniform ladder AB, of mass m and length $2a$

The ladder and the child are in equilibrium. By modelling the ladder as a rod and the child as a particle, calculate the least possible value of the coefficient of friction between the ladder and the ground.

Solution

A: F N be the frictional force and R N be the normal reaction.
B: P N be the normal reaction.

$$\begin{aligned} R(\updownarrow): &\quad R = 2mg + mg \\ R(\leftrightarrow): &\quad F = P \\ \text{Limiting equilibrium}: &\quad F \leqslant \mu R \\ \text{Moments about } A: &\quad (\tfrac{1}{2}a)(1.2mg) + a(0.6mg) = (2a)(0.8P). \end{aligned}$$

Now,

$$\begin{aligned} & (\tfrac{1}{2}a)(1.2mg) + a(0.6mg) = (2a)(0.8P) \\ \Rightarrow\ & 1.2mg = 1.6P \\ \Rightarrow\ & 1.2mg = 1.6F \\ \Rightarrow\ & F = 0.75mg \\ \Rightarrow\ & 0.75mg \leqslant \mu R \\ \Rightarrow\ & 0.75mg \leqslant 3\mu mg \\ \Rightarrow\ & \underline{\underline{\mu \geqslant 0.25}}. \end{aligned}$$

2. A uniform ladder, of weight W and length $2a$, rests in equilibrium with one end A on a smooth horizontal floor and the other end B on a rough vertical wall. The ladder is in a vertical plane perpendicular to the wall. The coefficient of friction between the wall and the ladder is μ. The ladder makes an angle θ with the floor, where $\tan\theta = 2$. A horizontal light inextensible string CD is attached to the ladder at the point C, where $AC = \frac{1}{2}a$. The string is attached to the wall at the point D, with BD vertical, as shown in Figure 2.

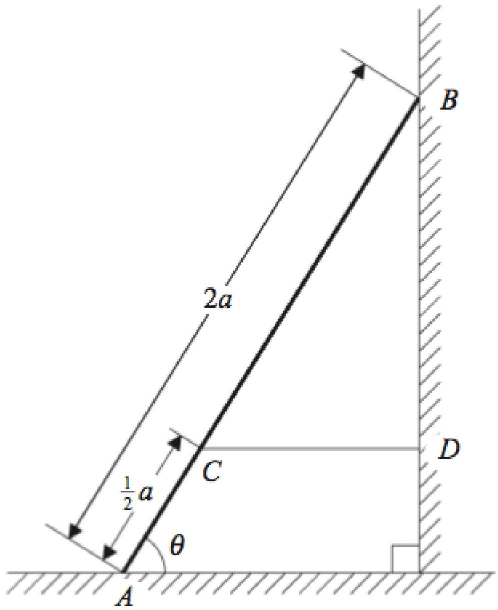


Figure 2: a uniform ladder, of weight W and length $2a$

The tension in the string is $\frac{1}{4}W$. By modelling the ladder as a rod,

(a) find the magnitude of the force of the floor on the ladder, (5)

Solution

A: P N be the normal reaction.
B: F N be the frictional force and R N be the normal reaction.
C: S N be the tension.

$$\begin{aligned} R(\updownarrow): &\quad W = F + P \\ R(\leftrightarrow): &\quad R = \tfrac{1}{4}W \\ \text{Equilibrium}: &\quad F \leqslant \mu R \\ \text{Moments about } B: &\quad P(2a\cos\theta) = W(a\cos\theta) + (\tfrac{1}{4}W)(\tfrac{3}{2}a\sin\theta). \end{aligned}$$

Now,

$$\begin{aligned} & P(2a\cos\theta) = W(a\cos\theta) + (\tfrac{1}{4}W)(\tfrac{3}{2}a\sin\theta) \\ \Rightarrow\ & 2P\cos\theta = W\cos\theta + \tfrac{3}{8}W\sin\theta \\ \Rightarrow\ & 2P \times \tfrac{1}{\sqrt{5}} = W \times \tfrac{1}{\sqrt{5}} + \tfrac{3}{8}W \times \tfrac{2}{\sqrt{5}} \\ \Rightarrow\ & 2P = W + \tfrac{3}{4}W \\ \Rightarrow\ & 2P = \tfrac{7}{4}W \\ \Rightarrow\ & \underline{\underline{P = \tfrac{7}{8}W}}. \end{aligned}$$

(b) show that $\mu \geqslant \frac{1}{2}$. (4)

Solution

$$W = F + P \Rightarrow W = F + \tfrac{7}{8}W$$
$$\Rightarrow \tfrac{1}{8}W \leqslant \tfrac{1}{4}\mu W$$
$$\Rightarrow \underline{\underline{\tfrac{1}{2} \leqslant \mu}}.$$

(c) State how you have used the modelling assumption that the ladder is a rod. (1)

Solution

E.g., <u>it does not bend</u> or <u>it negligible thickness</u>.

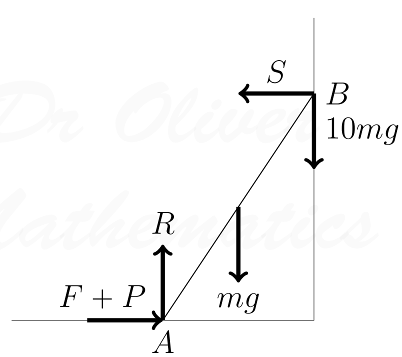
3. A uniform ladder AB, of mass m and length $2a$, has one end A on rough horizontal ground. The coefficient of friction between the ladder and the ground is 0.6. The other end B of the ladder rests against a smooth vertical wall.

A builder of mass $10m$ stands at the top of the ladder. To prevent the ladder from slipping, the builders friend pushes the bottom of the ladder horizontally towards the wall with a force of magnitude P. This force acts in a direction perpendicular to the wall. The ladder rests in equilibrium in a vertical plane perpendicular to the wall and makes an angle α with the horizontal, where $\tan \alpha = \frac{3}{2}$.

(a) Show that the reaction of the wall on the ladder has magnitude $7mg$. (5)

Solution

Let's start with a picture.



A: F N be the frictional force, R N be the normal reaction, and P N be the friend's push.

B: S N be the normal reaction.

$$\begin{aligned} R(\updownarrow): &\quad R = mg + 10mg \\ R(\leftrightarrow): &\quad F + P = S \\ \text{Limiting equilibrium}: &\quad F = 0.6R \\ \text{Moments about } A: &\quad mg(a\cos\alpha) + 10mg(2a\cos\alpha) = S(2a\sin\alpha). \end{aligned}$$

Now,

$$\begin{aligned} & mg(a\cos\alpha) + 10m(2a\cos\alpha) = S(2a\sin\alpha) \\ \Rightarrow\quad & amg + 20amg = 2aS\tan\alpha \\ \Rightarrow\quad & 3S = 21mg \\ \Rightarrow\quad & \underline{\underline{S = 7mg}}. \end{aligned}$$

(b) Find, in terms of m and g, the range of values of P for which the ladder remains in equilibrium. (7)

Solution

$$R = 11mg$$

and

$$F = 0.6 \times 11mg = 6.6mg.$$

Now, for the minimum of P, we use

$$P = 7mg - 6.6mg = \underline{\underline{0.4mg}}$$

and, for the maximum of P, we use

$$P = 7mg + 6.6mg = \underline{\underline{13.6mg.}}$$

4. A uniform rod AB, of length $8a$ and weight W, is free to rotate in a vertical plane about a smooth pivot at A. One end of a light inextensible string is attached to B. The other end is attached to point C which is vertically above A, with $AC = 6a$. The rod is in equilibrium with AB horizontal, as shown in Figure 3.

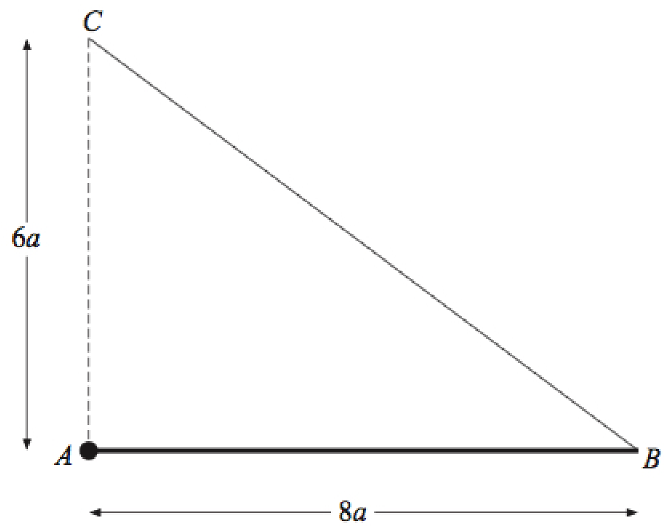
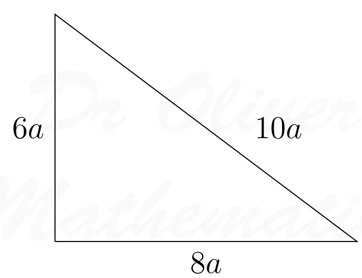


Figure 3: a uniform rod AB, of length $8a$ and weight W

(a) By taking moments about A, or otherwise, show that the tension in the string is $\frac{5}{6}W$. (4)

Solution



$\sin\alpha = \frac{3}{5}$

$\cos\alpha = \frac{4}{5}$

$$R(\updownarrow): \quad W = \tfrac{3}{5}R$$
$$\text{Moments about } A: \quad 4a \times W = 8a \times (\tfrac{3}{5}R).$$

Moments about A:

$$4a \times W = 8a \times (\tfrac{3}{5}R) \Rightarrow \underline{\underline{R = \tfrac{5}{6}W}}.$$

(b) Calculate the magnitude of the horizontal component of the force exerted by the pivot on the rod. (3)

Solution

Let S N be the horizontal component of the force exerted by the pivot on the rod. Then

$$S = \tfrac{5}{6}W \times \tfrac{4}{5} = \underline{\underline{\tfrac{2}{3}W}}.$$

5. A uniform pole AB, of mass 30 kg and length 3 m, is smoothly hinged to a vertical wall at one end A. The pole is held in equilibrium in a horizontal position by a light rod CD. One end C of the rod is fixed to the wall vertically below A. The other end D is freely jointed to the pole so that $\angle ACD = 30°$ and $AD = 0.5$ m, as shown in Figure 4.

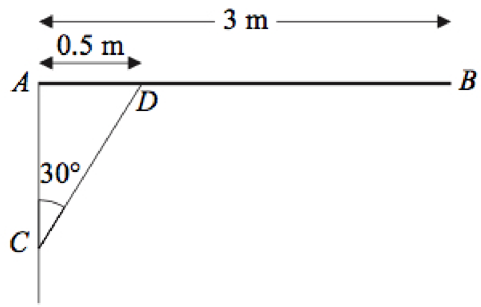


Figure 4: a uniform pole AB of mass 30 kg and length 3 m

Find

(a) the thrust in the rod CD, (4)

Solution

A: R N be the tension to the left and let S N be the tension to the down.
CD: T N be the tension.

$$\begin{aligned} R(\updownarrow): &\quad R = T\sin 30° \\ R(\leftrightarrow): &\quad T\cos 30° = S + 30g \\ \text{Moments about } A: &\quad T \times 0.5\cos 30° = 1.5 \times 30g. \end{aligned}$$

Moments about A:

$$\begin{aligned} T \times 0.5\cos 30° = 1.5 \times 30g &\Rightarrow T = \frac{45g}{0.5\cos 30°} \\ &\Rightarrow T = 60\sqrt{3}g \\ &\Rightarrow T = 1\,018.445\,875 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{T = 1\,000 \text{ (2 sf)}}}. \end{aligned}$$

(b) the magnitude of the force exerted by the wall on the pole at A. (6)

Solution

$$R = T \sin 30^\circ = 509.222\,937\,4 \text{ (FCD)}$$

and

$$S = T \cos 30^\circ - 30 = 588.$$

Now,

$$\begin{aligned} \text{resultant} &= \sqrt{509.222\ldots^2 + 588^2} \\ &= 777.850\,885\,5 \text{ (FCD)} \\ &= \underline{\underline{780 \text{ N (2 sf)}}}. \end{aligned}$$

The rod CD is removed and replaced by a longer light rod CM, where M is the mid-point of AB. The rod is freely jointed to the pole at M. The pole AB remains in equilibrium in a horizontal position.

(c) Show that the force exerted by the wall on the pole at A now acts horizontally. (2)

Solution

T and the mass of the object meet at mid-point M. In equilibrium, all forces act through a point. Hence <u><u>the reaction is horizontal</u></u>.

6. A ladder AB, of weight W and length $4a$, has one end A on rough horizontal ground. The coefficient of friction between the ladder and the ground is μ. The other end B rests against a smooth vertical wall. The ladder makes an angle θ with the horizontal, where $\tan\theta = 2$. A load of weight $4W$ is placed at the point C on the ladder, where $AC = 3a$, as shown in Figure 5.

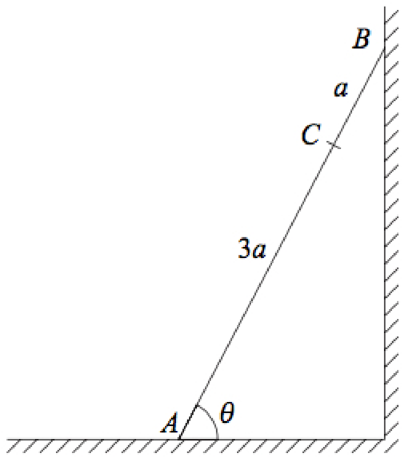


Figure 5: a ladder AB of weight W and length $4a$

The ladder is modelled as a uniform rod which is in a vertical plane perpendicular to the wall. The load is modelled as a particle. Given that the system is in limiting equilibrium,

(a) show that $\mu = 0.35$. (6)

Solution

A: F N be the frictional force and R N be the normal reaction.
B: P N be the normal reaction.

$$\begin{aligned} R(\updownarrow): &\quad R = W + 4W \\ R(\leftrightarrow): &\quad F = P \\ \text{Limiting equilibrium}: &\quad F = \mu R \\ \text{Moments about } A: &\quad W(2a\cos\theta) + 4W(3a\cos\theta) = P(4a\sin\theta). \end{aligned}$$

Now,

$$\begin{aligned} & W(2a\cos\theta) + 4W(3a\cos\theta) = P(4a\sin\theta) \\ \Rightarrow\ & 14W = 4P\tan\theta \\ \Rightarrow\ & 14W = 4\mu R \times 2 \\ \Rightarrow\ & 14W = 8\mu \times 5W \\ \Rightarrow\ & \underline{\underline{\mu = 0.35}}, \end{aligned}$$

as required.

A second load of weight kW is now placed on the ladder at A. The load of weight $4W$ is removed from C and placed on the ladder at B. The ladder is modelled as a uniform rod which is in a vertical plane perpendicular to the wall. The loads are modelled as particles. Given that the ladder and the loads are in equilibrium,

(b) find the range of possible values of k. (7)

Solution

$$\begin{aligned} R(\updownarrow): &\quad R = W + 4W + kW \\ R(\leftrightarrow): &\quad F = P \\ \text{Limiting equilibrium}: &\quad F \leqslant 0.35R \\ \text{Moments about } A: &\quad W(2a\cos\theta) + 4W(4a\cos\theta) = P(4a\sin\theta). \end{aligned}$$

Now,

$$\begin{aligned} & W(2a\cos\theta) + 4W(4a\cos\theta) = P(4a\sin\theta) \\ \Rightarrow\ & 18W = 4P\tan\theta \\ \Rightarrow\ & 18W = 4F \times 2 \\ \Rightarrow\ & 18W \leqslant 8 \times 0.35R \\ \Rightarrow\ & 18W \leqslant 2.4(5+k)W \\ \Rightarrow\ & 5 + k \geqslant \tfrac{45}{7} \\ \Rightarrow\ & k \geqslant \tfrac{10}{7} \\ \Rightarrow\ & \underline{\underline{k \geqslant 1.4 \text{ (2 sf)}}}. \end{aligned}$$

7. A wooden plank AB has mass $4m$ and length $4a$. The end A of the plank lies on rough horizontal ground. A small stone of mass m is attached to the plank at B. The plank is resting on a small smooth horizontal peg C, where $BC = a$, as shown in Figure 6.

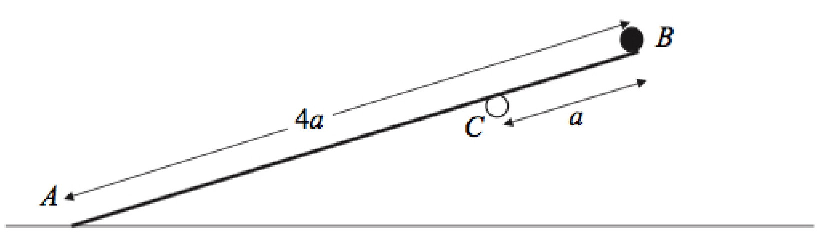


Figure 6: a wooden plank AB has mass $4m$ and length $4a$

The plank is in equilibrium making an angle α with the horizontal, where $\tan\alpha = \frac{3}{4}$. The coefficient of friction between the plank and the ground is μ. The plank is modelled as a uniform rod lying in a vertical plane perpendicular to the peg, and the stone as a particle. Show that

(a) the reaction of the peg on the plank has magnitude $\frac{16}{5}mg$, (3)

Solution

A: F N be the frictional force and R N be the normal reaction.
B: P N be the normal reaction.
C: Q N be the normal reaction.

$$\begin{aligned} R(\updownarrow): &\quad R + Q\cos\alpha = 4mg + mg \\ R(\leftrightarrow): &\quad F = Q\sin\alpha \\ \text{Limiting equilibrium}: &\quad F \leqslant \mu R \\ \text{Moments about } A: &\quad 3aQ = (2a)(4mg\cos\alpha) + (4a)(mg\cos\alpha). \end{aligned}$$

Now,

$$\begin{aligned} & 3aQ = (2a)(4mg\cos\alpha) + (4a)(mg\cos\alpha) \\ \Rightarrow\ & 3Q = 12mg \times \tfrac{4}{5} \\ \Rightarrow\ & \underline{\underline{Q = \tfrac{16}{5}mg}}. \end{aligned}$$

(b) $\mu \geqslant \frac{48}{61}$. (6)

Solution

$$\begin{aligned} R + Q\cos\alpha = 4mg + mg \Rightarrow R &= 5mg - \tfrac{16}{5}mg \times \tfrac{4}{5} \\ \Rightarrow R &= \tfrac{61}{25}mg, \end{aligned}$$

$$\begin{aligned} F &= Q\sin\alpha \\ &= \tfrac{16}{5}mg \times \tfrac{3}{5} \\ &= \tfrac{48}{25}mg, \end{aligned}$$

and

$$\begin{aligned} F \leqslant \mu R &\Rightarrow \tfrac{48}{25}mg \leqslant \tfrac{61}{25}\mu mg \\ &\Rightarrow \underline{\underline{\tfrac{48}{61} \leqslant \mu}}. \end{aligned}$$

(c) State how you have used the information that the peg is smooth. (1)

Solution
E.g., there is <u>no friction at the peg</u>.

8. A horizontal uniform rod AB has mass m and length $4a$. The end A rests against a rough vertical wall. A particle of mass $2m$ is attached to the rod at the point C, where $AC = 3a$. One end of a light inextensible string BD is attached to the rod at B and the other end is attached to the wall at a point D, where D is vertically above A. The rod is in equilibrium in a vertical plane perpendicular to the wall. The string is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$, as shown in Figure 7.

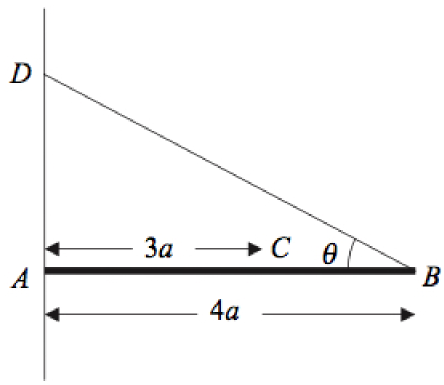


Figure 7: a horizontal uniform rod AB has mass m and length $4a$

(a) Find the tension in the string. (5)

Solution
A: F N be the frictional force and R N be the normal reaction.
B: P N be the tension.

$$\begin{aligned} R(\updownarrow): &\quad F + P\sin\theta = mg + 2mg \\ R(\leftrightarrow): &\quad R = P\cos\theta \\ \text{Limiting equilibrium}: &\quad F = \mu R \\ \text{Moments about } A: &\quad (2a)(mg) + (3a)(2mg) = (4a)(P\sin\theta). \end{aligned}$$

Now,

$$\begin{aligned} & (2a)(mg) + (3a)(2mg) = (4a)(P\sin\theta) \\ \Rightarrow\ & 8mg = 4P \times \tfrac{3}{5} \\ \Rightarrow\ & \underline{\underline{P = \tfrac{10}{3}mg}}. \end{aligned}$$

(b) Show that the horizontal component of the force exerted by the wall on the rod has magnitude $\frac{8}{3}mg$. (3)

Solution

$$\begin{aligned} R &= P\cos\theta \\ &= \tfrac{10}{3}mg \times \tfrac{4}{5} \\ &= \underline{\underline{\tfrac{8}{3}mg}}. \end{aligned}$$

The coefficient of friction between the wall and the rod is μ. Given that the rod is in limiting equilibrium,

(c) find the value of μ. (4)

Solution

$$F = 3mg - (\tfrac{10}{3}mg \times \tfrac{3}{5}) = mg$$

and

$$\begin{aligned} F = \mu R &\Rightarrow mg = \tfrac{8}{3}\mu mg \\ &\Rightarrow \underline{\underline{\mu = \tfrac{3}{8}}}. \end{aligned}$$

9. A uniform beam AB of mass 2 kg is freely hinged at one end A to a vertical wall. The beam is held in equilibrium in a horizontal position by a rope which is attached to a point C on the beam, where $AC = 0.14$ m. The rope is attached to the point D on the wall vertically above A, where $\angle ACD = 30°$, as shown in Figure 8.

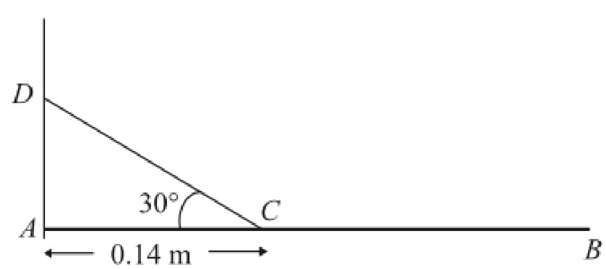


Figure 8: a uniform beam AB of mass 2 kg

The beam is modelled as a uniform rod and the rope as a light inextensible string. The tension in the rope is 63 N.

(a) the length of AB, (4)

Solution

A: F N be the frictional force and R N be the normal reaction.
B: P N be the tension.
Let x m be the length of AB.

$$\begin{aligned} R(\updownarrow): &\quad F + 63\sin 30^\circ = 2g \\ R(\leftrightarrow): &\quad R = 63\cos 30^\circ \\ \text{Limiting equilibrium}: &\quad F = \mu R \\ \text{Moments about } A: &\quad (0.14)(63\sin 30^\circ) = (\tfrac{1}{2}x)(2g). \end{aligned}$$

Now,

$$\begin{aligned} & (0.14)(63\sin 30^\circ) = (\tfrac{1}{2}x)(2g) \\ \Rightarrow\quad & 4.41 = 9.8x \\ \Rightarrow\quad & \underline{\underline{x = 0.45\text{ m}}}. \end{aligned}$$

(b) the magnitude of the resultant reaction of the hinge on the beam at A. (5)

Solution

$$F = 2g - 63\sin 30^\circ,$$
$$R = 63\cos 30^\circ,$$

and

$$\begin{aligned} \text{resultant} &= \sqrt{(2g - 63\sin 30^\circ)^2 + (63\cos 30^\circ)^2} \\ &= 55.842\,277\,89 \text{ (FCD)} \\ &= \underline{\underline{56\text{ N (2 sf)}}}. \end{aligned}$$

10. A ladder AB, of mass m and length $4a$, has one end A resting on rough horizontal ground. The other end B rests against a smooth vertical wall. A load of mass $3m$ is fixed on the ladder at the point C, where $AC = a$. The ladder is modelled as a uniform rod in a vertical plane perpendicular to the wall and the load is modelled as a particle. The ladder rests in limiting equilibrium making an angle of 30° with the wall, as shown in Figure 9.

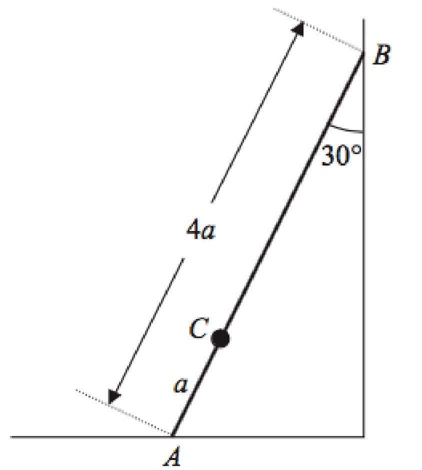


Figure 9: a ladder AB of mass m and length $4a$

Find the coefficient of friction between the ladder and the ground.

Solution

A: F N be the frictional force and R N be the normal reaction.
B: P N be the normal reaction.

$$\begin{aligned} R(\updownarrow): &\quad R = mg + 3mg \\ R(\leftrightarrow): &\quad F = P \\ \text{Limiting equilibrium}: &\quad F = \mu R \\ \text{Moments about } A: &\quad (3mg)(a\cos 60^\circ) + (mg)(2a\cos 60^\circ) = P(4a\sin 60^\circ). \end{aligned}$$

$$\begin{aligned} & (3mg)(a\cos 60^\circ) + (mg)(2a\cos 60^\circ) = P(4a\sin 60^\circ) \\ \Rightarrow\quad & 5mg = 4P\tan 60^\circ \\ \Rightarrow\quad & P = \tfrac{5\sqrt{3}}{12}mg. \end{aligned}$$

Finally,

$$\begin{aligned} F = \mu R &\Rightarrow \tfrac{5\sqrt{3}}{12}mg = 4mg\mu \\ &\Rightarrow \mu = \tfrac{5\sqrt{3}}{48} \\ &\Rightarrow \underline{\underline{\mu = 0.18 \text{ (2 sf)}}}. \end{aligned}$$

11. A plank rests in equilibrium against a fixed horizontal pole. The plank is modelled as a

uniform rod AB and the pole as a smooth horizontal peg perpendicular to the vertical plane containing AB. The rod has length $3a$ and weight W and rests on the peg at C, where $AC = 2a$. The end A of the rod rests on rough horizontal ground and AB makes an angle α with the ground, as shown in Figure 10.

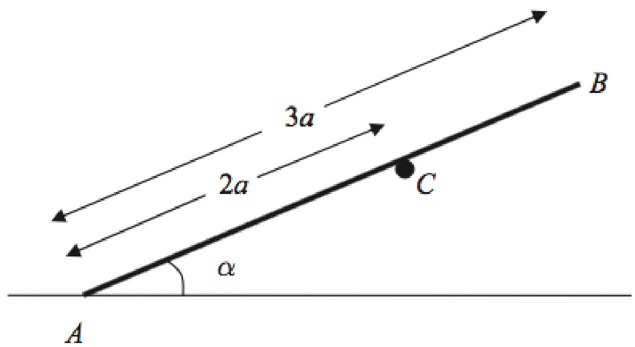


Figure 10: a plank rests in equilibrium against a fixed horizontal pole

(a) Show that the normal reaction on the rod at A is (6)

$$\tfrac{1}{4}(4 - 3\cos^2\alpha)W.$$

Solution

A: F N be the frictional force and R N be the normal reaction.
C: Q N be the normal reaction.

$$\begin{aligned} R(\updownarrow):&\quad R + Q\cos\alpha = W \\ R(\leftrightarrow):&\quad F = Q\sin\alpha \\ \text{Limiting equilibrium}:&\quad F = \mu R \\ \text{Moments about } A:&\quad (\tfrac{3}{2}a)(W\cos\alpha) = (2a)Q. \end{aligned}$$

Now,

$$(\tfrac{3}{2}a)(W\cos\alpha) = (2a)Q \Rightarrow Q = \tfrac{3}{4}W\cos\alpha$$

and

$$\begin{aligned} R &= W - Q\cos\alpha \\ &= W - \cos\alpha(\tfrac{3}{4}W\cos\alpha) \\ &= W - \tfrac{3}{4}W\cos^2\alpha \\ &= \underline{\underline{\tfrac{1}{4}(4 - 3\cos^2\alpha)W}}, \end{aligned}$$

as required.

Given that the rod is in limiting equilibrium and that $\cos\alpha = \frac{2}{3}$,

(b) find the coefficient of friction between the rod and the ground. (5)

Solution

$$\cos\alpha = \tfrac{2}{3} \Rightarrow R = \tfrac{2}{3}W$$

and

$$\begin{aligned} F &= Q\sin\alpha \\ &= \tfrac{3}{4}W\cos\alpha\sin\alpha \\ &= \tfrac{3}{4}W \times \tfrac{2}{3} \times \tfrac{\sqrt{5}}{3} \\ &= \tfrac{\sqrt{5}}{6}W. \end{aligned}$$

Finally,

$$\begin{aligned} F = \mu R \Rightarrow \mu &= \frac{\frac{\sqrt{5}}{6}W}{\frac{2}{3}W} \\ \Rightarrow \mu &= \tfrac{\sqrt{5}}{4} \\ \Rightarrow \underline{\underline{\mu = 0.56 \text{ (2 sf)}}}. \end{aligned}$$

12. Figure 11 shows a ladder AB, of mass 25 kg and length 4 m, resting in equilibrium with one end A on rough horizontal ground and the other end B against a smooth vertical wall. The ladder is in a vertical plane perpendicular to the wall. The coefficient of friction between the ladder and the ground is $\frac{11}{25}$. The ladder makes an angle β with the ground. When Reece, who has mass 75 kg, stands at the point C on the ladder, where $AC = 2.8$ m, the ladder is on the point of slipping.

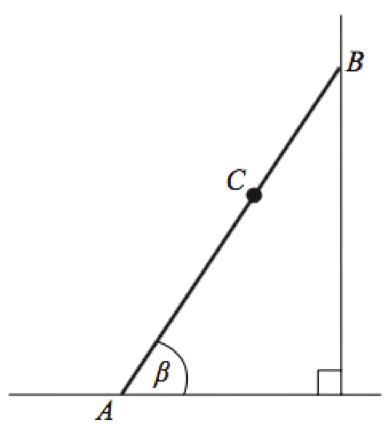


Figure 11: a ladder AB of mass 25 kg and length 4 m

The ladder is modelled as a uniform rod and Reece is modelled as a particle.

(a) Find the magnitude of the frictional force of the ground on the ladder. (3)

Solution

A: F N be the frictional force and R N be the normal reaction.
B: P N be the normal reaction.

$$\begin{aligned} R(\updownarrow): &\quad R = 25g + 75g \\ R(\leftrightarrow): &\quad F = P \\ \text{Limiting equilibrium}: &\quad F = \tfrac{11}{25}R \\ \text{Moments about } A: &\quad (2)(25g\cos\beta) + (2.8)(75g\cos\beta) = (4)(P\sin\beta). \end{aligned}$$

Now,

$$\begin{aligned} F &= \tfrac{11}{25}R \\ &= \tfrac{11}{25} \times 100g \\ &= 44g \\ &= \underline{\underline{430 \text{ N (2 sf)}}}. \end{aligned}$$

(b) Find, to the nearest degree, the value of β. (6)

Solution

$$\begin{aligned} & (2)(25g\cos\beta) + (2.8)(75g\cos\beta) = (4)(P\sin\beta) \\ \Rightarrow\ & 260g = 4 \times 44g\tan\beta \\ \Rightarrow\ & \tan\beta = \tfrac{65}{44} \\ \Rightarrow\ & \beta = 55.905\,022\,05 \text{ (FCD)} \\ \Rightarrow\ & \underline{\underline{\beta = 56^\circ \text{ (2 sf)}}} \end{aligned}$$

(c) State how you have used the modelling assumption that Reece is a particle. (1)

Solution

E.g., <u>his mass acts directly at the point C</u>.

13. A uniform rod AB, of length 1.5 m and mass 3 kg, is smoothly hinged to a vertical wall at A. The rod is held in equilibrium in a horizontal position by a light strut CD as shown in Figure 12.

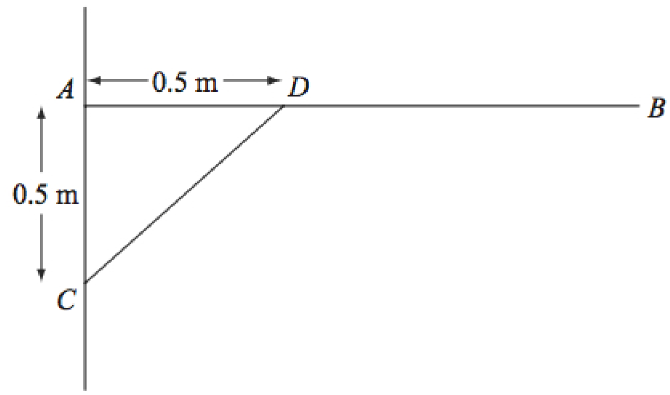


Figure 12: a uniform rod AB of length 1.5 m and mass 3 kg

The rod and the strut lie in the same vertical plane, which is perpendicular to the wall. The end C of the strut is freely jointed to the wall at a point 0.5 m vertically below A. The end D is freely joined to the rod so that AD is 0.5 m.

(a) Find the thrust in CD. (4)

Solution

A: R N be the tension to the left and let S N be the tension to the down.
CD: T N be the tension.

$$\begin{aligned} R(\updownarrow): &\quad R = T\sin 45^\circ \\ R(\leftrightarrow): &\quad T\cos 45^\circ = S + 3g \\ \text{Moments about } A: &\quad T \times 0.5\cos 45^\circ = 0.75 \times 3g. \end{aligned}$$

Now,

$$\begin{aligned} & T \times 0.5\cos 45^\circ = 0.75 \times 3g \\ \Rightarrow\quad & T = \frac{2.25g}{0.5\cos 45^\circ} \\ \Rightarrow\quad & T = \tfrac{9\sqrt{2}g}{2} \\ \Rightarrow\quad & \underline{\underline{T = 62 \text{ N (2 sf)}}}. \end{aligned}$$

(b) Find the magnitude and direction of the force exerted on the rod AB at A. (7)

Solution

$$R = T\sin 45^\circ = \tfrac{9g}{2}$$

and

$$S = T\cos 45^\circ - 3g = \tfrac{3g}{2}.$$

Now,

$$\begin{aligned}\text{resultant} &= \sqrt{(\tfrac{9g}{2})^2 + (\tfrac{3g}{2})^2}\\ &= \tfrac{3\sqrt{10}g}{2}\\ &= \underline{\underline{46\text{ N (2 sf)}}},\end{aligned}$$

at an angle

$$\begin{aligned}\tan^{-1}\frac{\frac{3g}{2}}{\frac{9g}{2}} &= \tan^{-1}\tfrac{1}{3}\\ &= 18.434\,948\,82\text{ (FCD)}\\ &= \underline{\underline{18^\circ\text{ (2 sf) below the line }BA.}}\end{aligned}$$

14. A uniform rod AB, of mass 20 kg and length 4 m, rests with one end A on rough horizontal ground. The rod is held in limiting equilibrium at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$, by a force acting at B, as shown in Figure 13. (7)

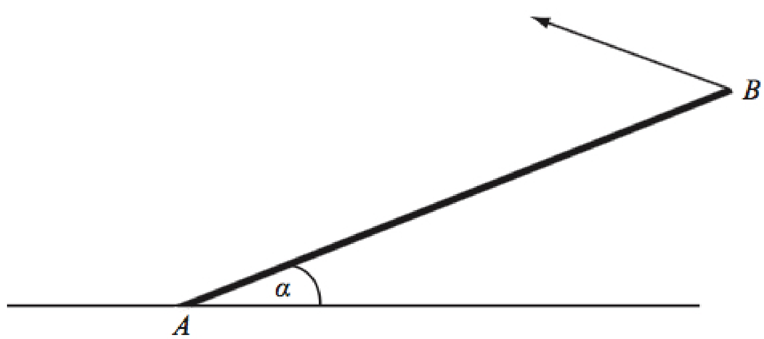


Figure 13: a uniform rod AB of mass 20 kg and length 4 m

The line of action of this force lies in the vertical plane which contains the rod. The coefficient of friction between the ground and the rod is 0.5. Find the magnitude of the normal reaction of the ground on the rod at A.

Solution

A: F N be the frictional force and R N be the normal reaction.

$$\text{Limiting equilibrium}: \quad F = 0.5R$$
$$\text{Moments about } B: \quad (4)(R\cos\alpha) = (2)(20g\cos\alpha) + (4)(F\sin\alpha).$$

Now,

$$\begin{aligned} & (4)(R\cos\alpha) = (2)(20g\cos\alpha) + (4)(F\sin\alpha) \\ \Rightarrow\ & 4R = 40g + 4F\tan\alpha \\ \Rightarrow\ & 4R = 40g + 3F \\ \Rightarrow\ & 4R = 40g + 1.5R \\ \Rightarrow\ & 2.5R = 40g \\ \Rightarrow\ & R = 16g \\ \Rightarrow\ & \underline{\underline{R = 160 \text{ N (2 sf)}}}. \end{aligned}$$

15. Figure 14 shows a uniform rod AB of mass m and length $4a$.

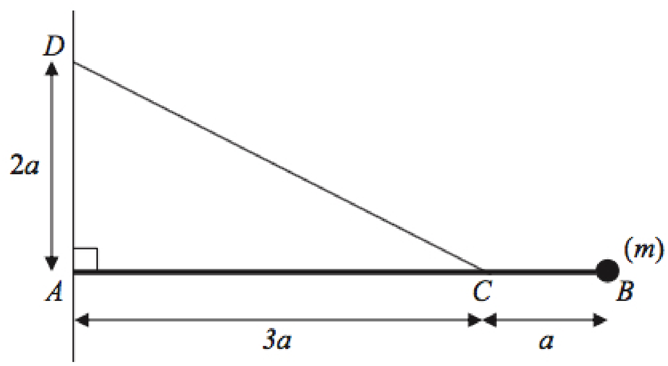


Figure 14: a uniform rod AB of mass m and length $4a$

The end A of the rod is freely hinged to a point on a vertical wall. A particle of mass m is attached to the rod at B. One end of a light inextensible string is attached to the rod at C, where $AC = 3a$. The other end of the string is attached to the wall at D, where $AD = 2a$ and D is vertically above A. The rod rests horizontally in equilibrium in a vertical plane perpendicular to the wall and the tension in the string is T.

(a) Show that $T = mg\sqrt{13}$. (5)

Solution

A: F N be the frictional force and R N be the normal reaction.
C: T N be the tension.

$$\text{Moments about } A: \quad (2a)(mg) + (4a)(mg) = (3a)(T\sin\theta).$$

Now,

$$\begin{aligned} & (2a)(mg) + (4a)(mg) = (3a)(T\sin\theta) \\ \Rightarrow\quad & 2mg = T \times \tfrac{2}{\sqrt{13}} \\ \Rightarrow\quad & \underline{\underline{T = mg\sqrt{13}}}. \end{aligned}$$

The particle of mass m at B is removed from the rod and replaced by a particle of mass M which is attached to the rod at B. The string breaks if the tension exceeds $2mg\sqrt{13}$. Given that the string does not break,

(b) show that $M \leqslant \frac{5}{2}m$. (3)

Solution

$$\text{Moments about } A: \quad (2a)(mg) + (4a)(Mg) \leqslant (3a)(2mg\sqrt{13}\sin\theta).$$

Now,

$$\begin{aligned} & (2a)(mg) + (4a)(Mg) \leqslant (3a)(2mg\sqrt{13}\sin\theta) \\ \Rightarrow\quad & m + 2M \leqslant 6m \\ \Rightarrow\quad & 2M \leqslant 5m \\ \Rightarrow\quad & \underline{\underline{M \leqslant \tfrac{5}{2}m}}. \end{aligned}$$

16. A uniform plank AB, of weight 100 N and length 4 m, rests in equilibrium with the end A on rough horizontal ground. The plank rests on a smooth cylindrical drum. The drum is fixed to the ground and cannot move. The point of contact between the plank and the drum is C, where $AC = 3$ m, as shown in Figure 15. (10)

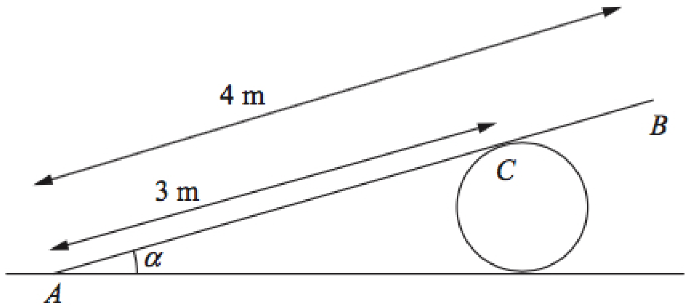


Figure 15: a uniform plank AB of weight 100 N and length 4 m

The plank is resting in a vertical plane which is perpendicular to the axis of the drum, at an angle α to the horizontal, where $\sin\alpha = \frac{1}{3}$. The coefficient of friction between the plank and the ground is μ. Modelling the plank as a rod, find the least possible value of μ.

Solution

A: F N be the frictional force and R N be the normal reaction.
C: Q N be the normal reaction.

$$\begin{aligned} R(\updownarrow):&\quad R + Q\cos\alpha = 100 \\ R(\leftrightarrow):&\quad F = Q\sin\alpha \\ \text{Limiting equilibrium}:&\quad F = \mu R \\ \text{Moments about } A:&\quad (2a)(100\cos\alpha) = (3a)Q. \end{aligned}$$

Now,

$$(2a)(100\cos\alpha) = (3a)Q \Rightarrow Q = \tfrac{200\sqrt{8}}{9},$$
$$F = Q\sin\alpha\tfrac{200\sqrt{8}}{27},$$

$$(2a)(100\cos\alpha) = (3a)Q \Rightarrow Q = \tfrac{200\sqrt{8}}{9}.$$
$$F = Q\sin\alpha = \tfrac{200\sqrt{8}}{27},$$

and

$$R = 100 - Q\cos\alpha = \tfrac{1100}{27}.$$

Finally,

$$\begin{aligned} F = \mu R &\Rightarrow \tfrac{200\sqrt{8}}{27} = \tfrac{1100}{27}\mu \\ &\Rightarrow \mu = \tfrac{4\sqrt{2}}{11} \\ &\Rightarrow \underline{\underline{\mu = 0.51 \text{ (2 sf)}}}. \end{aligned}$$

17. A uniform rod AB, of mass $3m$ and length $4a$, is held in a horizontal position with the end A against a rough vertical wall. One end of a light inextensible string BD is attached to the rod at B and the other end of the string is attached to the wall at the point D vertically above A, where $AD = 3a$. A particle of mass $3m$ is attached to the rod at C, where $AC = x$. The rod is in equilibrium in a vertical plane perpendicular to the wall as shown in Figure 16.

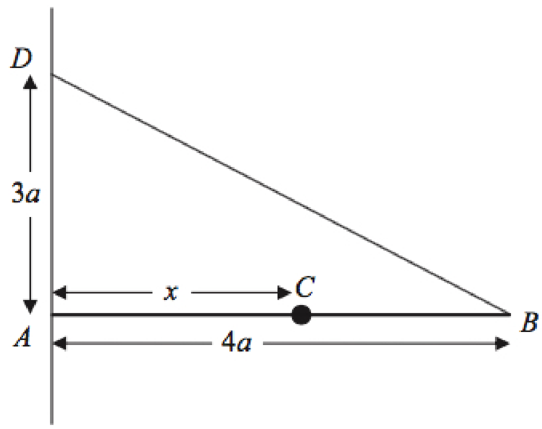


Figure 16: a uniform rod AB of mass $3m$ and length $4a$

The tension in the string is $\frac{25}{4}mg$. Show that

(a) $x = 3a$, (5)

Solution

A: F N be the frictional force and R N be the normal reaction.

$$\begin{aligned} R(\updownarrow):&\quad F + \tfrac{25}{4}mg\sin\theta = 3mg + 3mg \\ R(\leftrightarrow):&\quad R = \tfrac{25}{4}mg\cos\theta \\ \text{Limiting equilibrium}:&\quad F = \mu R \\ \text{Moments about } A:&\quad (2a)(3mg) + (x)(3mg) = (4a)(\tfrac{25}{4}mg\sin\theta). \end{aligned}$$

Now,

$$\begin{aligned} & (2a)(3mg) + (x)(3mg) = (4a)(\tfrac{25}{4}mg\sin\theta) \\ \Rightarrow\ & 6a + 3x = 4a \times \tfrac{25}{4}mg \times \tfrac{3}{5} \\ \Rightarrow\ & 6a + 3x = 9a \\ \Rightarrow\ & \underline{\underline{x = 3a}}, \end{aligned}$$

as required.

(b) the horizontal component of the force exerted by the wall on the rod has magnitude $5mg$. (3)

> **Solution**
>
> $$R = \tfrac{25}{4}mg \times \tfrac{4}{5} = \underline{\underline{5mg}},$$
>
> as required.

The coefficient of friction between the wall and the rod is μ. Given that the rod is about to slip,

(c) find the value of μ. (5)

> **Solution**
>
> $$F = 6mg - (\tfrac{25}{4}mg \times \tfrac{3}{5}) = \tfrac{9}{4}mg$$
>
> and
>
> $$\tfrac{9}{4}mg = 5mg\mu \Rightarrow \underline{\underline{\mu = 0.45}}.$$

18. A uniform rod AB has mass 4 kg and length 1.4 m. The end A is resting on rough horizontal ground. A light string BC has one end attached to B and the other end attached to a fixed point C. The string is perpendicular to the rod and lies in the same vertical plane as the rod. The rod is in equilibrium, inclined at 20° to the ground, as shown in Figure 17.

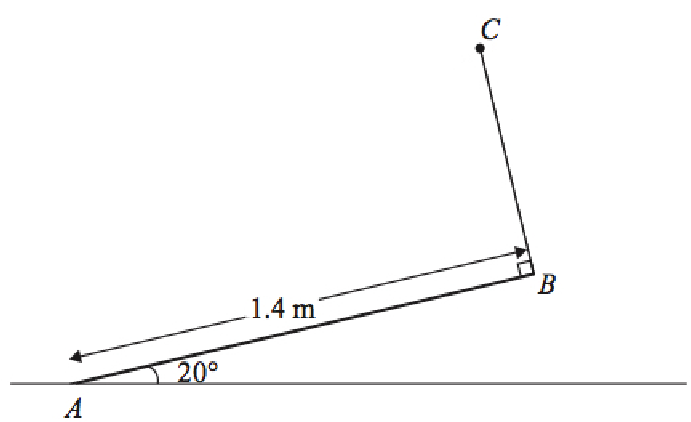


Figure 17: a uniform rod AB has mass 4 kg and length 1.4 m

(a) Find the tension in the string. (4)

Solution

A: F N be the frictional force and R N be the normal reaction.
C: Q N be the normal reaction.

$$\begin{aligned} R(\updownarrow):& \quad R + Q\cos 20^\circ = 4g \\ R(\leftrightarrow):& \quad F = Q\sin 20^\circ \\ \text{Limiting equilibrium}:& \quad F = \mu R \\ \text{Moments about } A:& \quad (2g)(0.7\cos 20^\circ) = 1.4Q. \end{aligned}$$

Now,

$$\begin{aligned} & (4g)(0.7\cos 20^\circ) = 1.4Q \\ \Rightarrow \quad & Q = 2g\cos 20^\circ \\ \Rightarrow \quad & \underline{\underline{Q = 18 \text{ N (2 sf)}}}. \end{aligned}$$

Given that the rod is about to slip,

(b) find the coefficient of friction between the rod and the ground. (7)

Solution

$$\begin{aligned} F = \mu R \Rightarrow 2g\sin 20^\circ\cos 20^\circ &= 2g(2-\cos^2 20^\circ)\mu \\ \Rightarrow \mu &= \frac{\sin 20^\circ\cos 20^\circ}{2-\cos^2 20^\circ} \\ \Rightarrow \mu &= 0.287\,735\,182\,4 \text{ (FCD)} \\ \Rightarrow \underline{\underline{\mu}} &\underline{\underline{= 0.29 \text{ (2 sf)}}}. \end{aligned}$$

19. A uniform rod AB, of mass 5 kg and length 4 m, has its end A smoothly hinged at a fixed point. The rod is held in equilibrium at an angle of 25° above the horizontal by a force of magnitude F newtons applied to its end B. The force acts in the vertical plane containing the rod and in a direction which makes an angle of 40° with the rod, as shown in Figure 18.

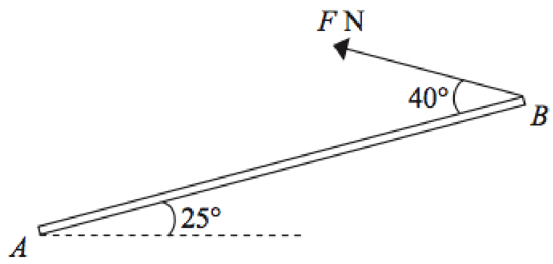


Figure 18: a uniform rod AB of mass 5 kg and length 4 m

(a) Find the value of F. (4)

Solution

A: R N be the normal reaction.
Note the F only makes an angle $40 - 25 = 15^\circ$ with the vertical.

$$R(\updownarrow): \quad R + F\sin 15^\circ = 5g$$
$$\text{Moments about } A: \quad (2)(5g\cos 25^\circ) = F(4\sin 40^\circ).$$

Now,

$$(2)(5g\cos 25^\circ) = F(4\sin 40^\circ) \Rightarrow F = \frac{10g\cos 25^\circ}{4\sin 40^\circ}$$
$$\Rightarrow F = 34.544\,133\,16 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{F = 35 \text{ N (2 sf)}}}.$$

(b) Find the magnitude and direction of the vertical component of the force acting on the rod at A. (4)

Solution

$$R = 5g - F\sin 15^\circ$$
$$= 40.059\,320\,44 \text{ (FCD)}$$
$$= \underline{\underline{40 \text{ N (2 sf) upwards}}}.$$

20. A ladder, of length 5 m and mass 18 kg, has one end A resting on rough horizontal ground and its other end B resting against a smooth vertical wall. The ladder lies in a vertical plane perpendicular to the wall and makes an angle α with the horizontal ground, where $\tan\alpha = \frac{4}{3}$, as shown in Figure 19. (9)

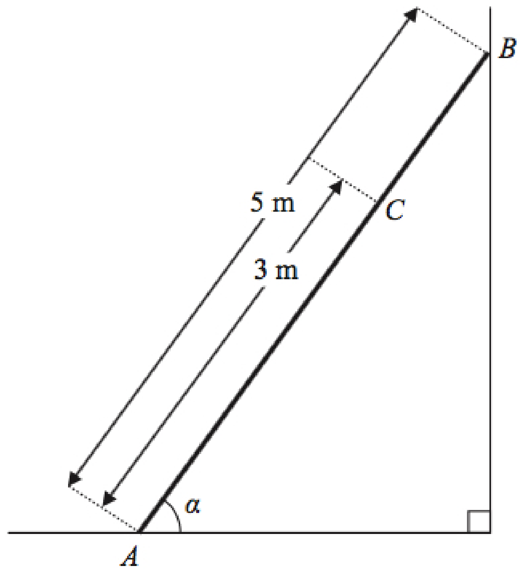


Figure 19: a ladder of length 5 m and mass 18 kg

The coefficient of friction between the ladder and the ground is μ. A woman of mass 60 kg stands on the ladder at the point C, where $AC = 3$ m. The ladder is on the point of slipping. The ladder is modelled as a uniform rod and the woman as a particle.

Find the value of μ.

Solution

A: F N be the frictional force and R N be the normal reaction.
B: P N be the normal reaction.

$$\begin{aligned} R(\updownarrow): &\quad R = 18g + 60g \\ R(\leftrightarrow): &\quad F = P \\ \text{Limiting equilibrium}: &\quad F = \mu R \\ \text{Moments about } A: &\quad 2.5(18g\cos\alpha) + 3(60g\cos\alpha) = 5(P\sin\alpha). \end{aligned}$$

Now,

$$\begin{aligned} & 2.5(18g\cos\alpha) + 3(60g\cos\alpha) = 5(P\sin\alpha) \\ \Rightarrow\quad & 225g = 5P\tan\alpha \\ \Rightarrow\quad & 225g = \tfrac{20}{3}P \\ \Rightarrow\quad & P = \tfrac{135}{4}g, \end{aligned}$$

$$F = \tfrac{135}{4}g \text{ and } R = 78g.$$

Finally,

$$\begin{aligned} F = \mu R &\Rightarrow \tfrac{135}{4}g = 78g\mu \\ &\Rightarrow \mu = \tfrac{45}{104} \\ &\Rightarrow \underline{\underline{\mu = 0.43 \text{ (2 sf)}}}. \end{aligned}$$

21. A uniform rod AB, of mass m and length $2a$, is freely hinged to a fixed point A. A particle of mass m is attached to the rod at B. The rod is held in equilibrium at an angle θ to the horizontal by a force of magnitude F acting at the point C on the rod, where $AC = b$, as shown in Figure 20.

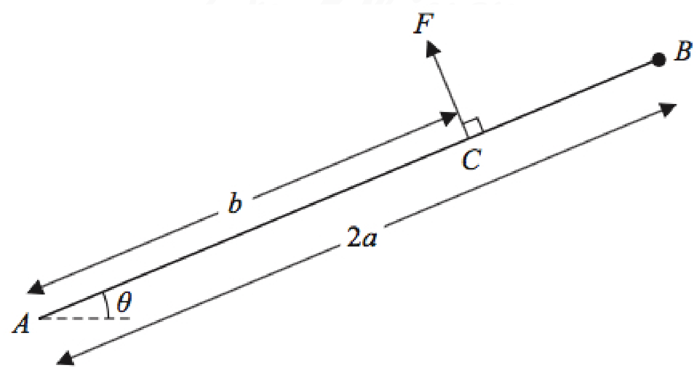


Figure 20: a uniform rod AB of mass m and length $2a$

The force at C acts at right angles to AB and in the vertical plane containing AB.

(a) Show that $F = \dfrac{3amg\cos\theta}{b}$. (4)

Solution

A: R N be the frictional force and S N be the normal reaction.
C: F N be the normal reaction.

$$\begin{aligned} R(\updownarrow): &\quad S + F\cos\theta = mg + mg \\ R(\leftrightarrow): &\quad R = F\sin\theta \\ \text{Limiting equilibrium}: &\quad F \leqslant \mu R \\ \text{Moments about } A: &\quad a(mg\cos\theta) + (2a)(mg\cos\theta) = bF. \end{aligned}$$

Now,

$$\begin{aligned} & a(mg\cos\theta) + (2a)(mg\cos\theta) = bF \\ \Rightarrow \; & bF = 3amg\cos\theta \\ \Rightarrow \; & \underline{\underline{F = \frac{3amg\cos\theta}{b}}}, \end{aligned}$$

as required.

(b) Find, in terms of a, b, g, m, and θ, (5)

(i) the horizontal component of the force acting on the rod at A,

Solution

$$R = F\sin\theta = \underline{\underline{\frac{3amg\sin\theta\cos\theta}{b}}}.$$

(ii) the vertical component of the force acting on the rod at A.

Solution

$$\begin{aligned} S &= 2mg - F\cos\theta \\ &= 2mg - \frac{3amg\cos^2\theta}{b} \\ &= \underline{\underline{\frac{mg(2b - 3a\cos^2\theta)}{b}}}. \end{aligned}$$

Given that the force acting on the rod at A acts along the rod,

(c) find the value of $\frac{a}{b}$. (4)

Solution

$$\tan\theta = \frac{\frac{mg(2b-3a\cos^2\theta)}{b}}{\frac{3amg\sin\theta\cos\theta}{b}}$$
$$\Rightarrow \quad \tan\theta = \frac{2b-3a\cos^2\theta}{3a\sin\theta\cos\theta}$$
$$\Rightarrow \quad 3a\sin^2\theta = 2b - 3a\cos^2\theta$$
$$\Rightarrow \quad 3a(\sin^2\theta + \cos^2\theta) = 2b$$
$$\Rightarrow \quad 3a = 2b$$
$$\Rightarrow \quad \underline{\underline{\tfrac{a}{b} = \tfrac{2}{3}}}.$$

22. A rough circular cylinder of radius $4a$ is fixed to a rough horizontal plane with its axis horizontal. A uniform rod AB, of weight W and length $6a\sqrt{3}$, rests with its lower end A on the plane and a point C of the rod against the cylinder. The vertical plane through the rod is perpendicular to the axis of the cylinder. The rod is inclined at 60° to the horizontal, as shown in Figure 21.

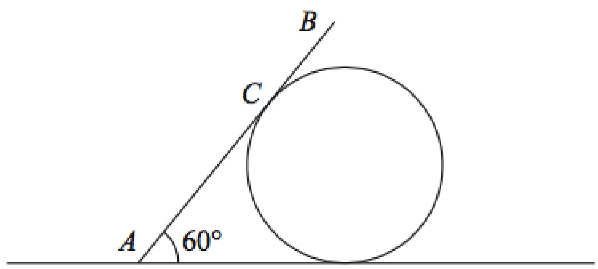
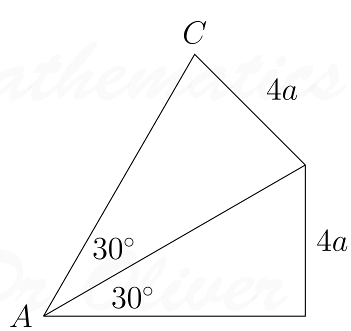


Figure 21: a uniform rod AB of weight W and length $6a\sqrt{3}$

(a) Show that $AC = 4a\sqrt{3}$. (2)

Solution



$$AC = \frac{4a}{\tan 30^\circ} = \underline{\underline{4a\sqrt{3}}}.$$

The coefficient of friction between the rod and the cylinder is $\frac{\sqrt{3}}{3}$ and the coefficient of friction between the rod and the plane is μ. Given that friction is limiting at both A and C,

(b) find the value of μ. (9)

Solution

You need the frictional and normal reaction *at both A and C*.
A: F_A N be the frictional force and R_A N be the normal reaction.
C: F_C N be the frictional force and R_C N be the normal reaction.

$$\begin{aligned}
R(\updownarrow): &\quad R_A + R_C\cos 60^\circ + F_C\cos 30^\circ = W \\
R(\leftrightarrow): &\quad F_A + F_C\cos 60^\circ = R_C\sin 60^\circ \\
\text{Limiting equilibrium at } A: &\quad F_A = \mu R_A \\
\text{Limiting equilibrium at } C: &\quad F_C = \tfrac{\sqrt{3}}{3}R_C \\
\text{Moments about } A: &\quad W(3a\sqrt{3}\cos 60^\circ) = (4a\sqrt{3})R_C.
\end{aligned}$$

Now,

$$\tfrac{3}{2}W = 4R_C \Rightarrow R_C = \tfrac{3}{8}W,$$

$$F_C = \tfrac{\sqrt{3}}{3} \times \tfrac{3}{8}W = \tfrac{3\sqrt{3}}{24}W,$$

$$F_A = R_C\sin 60^\circ - F_C\cos 60^\circ = \tfrac{\sqrt{3}}{8}W,$$

and

$$R_A = W - R_C\cos 60^\circ - F_C\cos 30^\circ = \tfrac{5}{8}W.$$

Finally,

$$\begin{aligned}
F_A = \mu R_A &\Rightarrow \mu = \frac{\frac{\sqrt{3}}{8}W}{\frac{5}{8}W} \\
&\Rightarrow \mu = \tfrac{\sqrt{3}}{5} \\
&\Rightarrow \underline{\underline{\mu = 0.35 \text{ (2 sf)}}}.
\end{aligned}$$

23. A uniform rod AB of weight W has its end A freely hinged to a point on a fixed vertical wall. The rod is held in equilibrium, at angle θ to the horizontal, by a force of magnitude P. The force acts perpendicular to the rod at B and in the same vertical plane as the

rod, as shown in Figure 22.

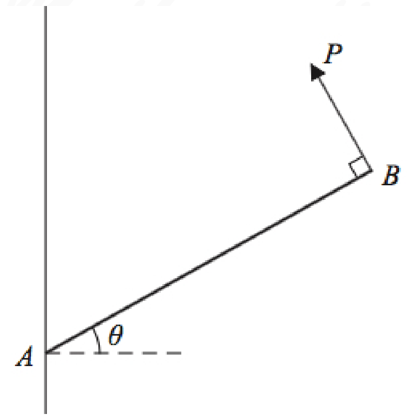


Figure 22: a uniform rod AB of weight W

The rod is in a vertical plane perpendicular to the wall. The magnitude of the vertical component of the force exerted on the rod by the wall at A is Y.

(a) Show that (6)

$$Y = \frac{W}{2}(2 - \cos^2\theta).$$

Solution

A: F N be the frictional force, X N be the horizontal normal reaction, and Y N be the vertical normal reaction.
B: P N be the normal reaction.
Let $2a$ be the length of the rod.

$$\begin{aligned} R(\updownarrow): &\quad Y + P\cos\theta = W \\ R(\leftrightarrow): &\quad X = P\sin\theta \\ \text{Moments about } A: &\quad a(W\cos\theta) = 2aP. \end{aligned}$$

Now,

$$\begin{aligned} W\cos\theta = 2P &\Rightarrow P = \tfrac{W}{2}\cos\theta \\ &\Rightarrow Y = W - P\cos\theta \\ &\Rightarrow Y = W - \tfrac{W}{2}\cos^2\theta \\ &\Rightarrow \underline{\underline{Y = \tfrac{W}{2}(2 - \cos^2\theta)}}. \end{aligned}$$

Given that $\theta = 45°$,

(b) find the magnitude of the force exerted on the rod by the wall at A, giving your answer in terms of W. (6)

Solution

$$X = \sin 45°(\tfrac{W}{2}\cos 45°) = \tfrac{1}{4}W$$

and

$$Y = \tfrac{W}{2}(2 - \cos^2 45°) = \tfrac{3}{4}W.$$

Finally,

$$\begin{aligned}\text{resultant} &= \sqrt{(\tfrac{3}{4}W)^2 + (\tfrac{1}{4}W)^2} \\ &= \underline{\underline{\tfrac{\sqrt{10}}{4}W}}.\end{aligned}$$

24. A non-uniform rod, AB, of mass m and length $2l$, rests in equilibrium with one end A on a rough horizontal floor and the other end B against a rough vertical wall. The rod is in a vertical plane perpendicular to the wall and makes an angle of $60°$ with the floor as shown in Figure 23.

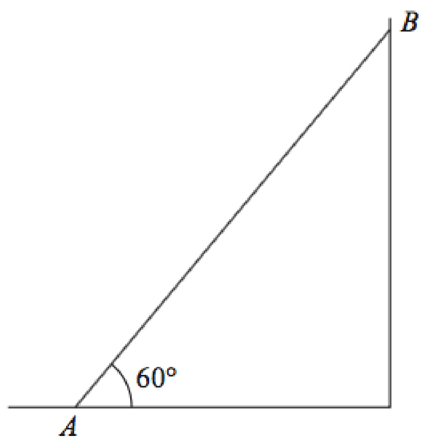


Figure 23: a non-uniform rod, AB, of mass m and length $2l$

The coefficient of friction between the rod and the floor is $\frac{1}{4}$ and the coefficient of friction between the rod and the wall is $\frac{2}{3}$. The rod is on the point of slipping at both ends.

(a) Find the magnitude of the vertical component of the force exerted on the rod by the floor. (5)

Solution

A: F_A N be the frictional force and R_A N be the normal reaction.
B: F_B N be the frictional force and R_B N be the normal reaction.

$$\begin{aligned} R(\updownarrow):&\quad R_A + F_B = mg \\ R(\leftrightarrow):&\quad F_A = R_B \\ \text{Limiting equilibrium at } A:&\quad F_A = \tfrac{1}{4}R_A \\ \text{Limiting equilibrium at } B:&\quad F_B = \tfrac{2}{3}R_B \\ \text{Moments about } A:&\quad F_B(2l\cos 60^\circ) + R_B(2l\sin 60^\circ) = AB(mg\cos 60^\circ). \end{aligned}$$

$$\begin{aligned} R_A = mg - F_B &\Rightarrow R_A = mg - \tfrac{2}{3}R_B \\ &\Rightarrow R_A = mg - \tfrac{2}{3}F_A \\ &\Rightarrow R_A = mg - \tfrac{2}{3}(\tfrac{1}{4}R_A) \\ &\Rightarrow R_A = mg - \tfrac{1}{6}R_A \\ &\Rightarrow \tfrac{7}{6}R_A = mg \\ &\Rightarrow \underline{\underline{R_A = \tfrac{6}{7}mg}}. \end{aligned}$$

The centre of mass of the rod is at G.

(b) Find the distance AG. (5)

Solution

$$\begin{aligned} F_A = \tfrac{1}{4}R_A &\Rightarrow F_A = \tfrac{1}{4}\times\tfrac{6}{7}mg \\ &\Rightarrow F_A = \tfrac{3}{14}mg \\ &\Rightarrow R_B = \tfrac{3}{14}mg \\ &\Rightarrow F_B = mg - \tfrac{6}{7}mg = \tfrac{1}{7}mg. \end{aligned}$$

Now,

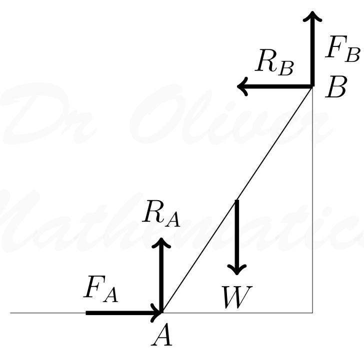
$$\begin{aligned} & F_B(2l\cos 60^\circ) + R_B(2l\sin 60^\circ) = AB(mg\cos 60^\circ) \\ \Rightarrow\quad & \tfrac{1}{7}(2l\cos 60^\circ) + \tfrac{3}{14}(2l\sin 60^\circ) = \tfrac{1}{2}AB \\ \Rightarrow\quad & AB = \tfrac{1}{7}l + \tfrac{3\sqrt{3}}{7}l \\ \Rightarrow\quad & AB = \tfrac{l(2+3\sqrt{3})}{7} \\ \Rightarrow\quad & \underline{\underline{AB = 1.0l\text{ m (2 sf)}}}. \end{aligned}$$

25. A ladder AB, of weight W and length $2l$, has one end A resting on rough horizontal ground. The other end B rests against a rough vertical wall. The coefficient of friction between the ladder and the wall is $\frac{1}{3}$. The coefficient of friction between the ladder and the ground is μ. Friction is limiting at both A and B. The ladder is at an angle θ to the ground, where $\tan\theta = \frac{5}{3}$. The ladder is modelled as a uniform rod which lies in a vertical plane perpendicular to the wall.

Find the value of μ.

Solution

Let's start with a picture, shall we?



A: F_A N be the frictional force and R_A N be the normal reaction.
B: F_B N be the frictional force and R_B N be the normal reaction.

$$\begin{aligned}
R(\updownarrow): &\quad R_A + F_B = W \\
R(\leftrightarrow): &\quad F_A = R_B \\
\text{Limiting equilibrium at } A: &\quad F_A = \mu R_A \\
\text{Limiting equilibrium at } B: &\quad F_B = \tfrac{1}{3} R_B \\
\text{Moments about } A: &\quad F_B(2l\cos\theta) + R_B(2l\sin\theta) = l(W\cos\theta).
\end{aligned}$$

$$
\begin{aligned}
& F_B(2l\cos\theta) + R_B(2l\sin\theta) = l(W\cos\theta) \\
\Rightarrow\quad & 2F_B + 2R_B\tan\theta = W \\
\Rightarrow\quad & 2F_B + \tfrac{10}{3}R_B = W \\
\Rightarrow\quad & 2F_B = W - \tfrac{10}{3}R_B \\
\Rightarrow\quad & F_B = \tfrac{1}{2}W - \tfrac{5}{3}R_B \\
\Rightarrow\quad & \tfrac{1}{3}R_B = \tfrac{1}{2}W - \tfrac{5}{3}R_B \\
\Rightarrow\quad & 2R_B = \tfrac{1}{2}W \\
\Rightarrow\quad & R_B = \tfrac{1}{4}W \\
\Rightarrow\quad & F_B = \tfrac{1}{12}W.
\end{aligned}
$$

Now,

$$F_A = R_B = \tfrac{1}{4}W$$

and

$$R_A = W - F_B = \tfrac{11}{12}W.$$

Finally,

$$
\begin{aligned}
\tfrac{1}{4}W = \tfrac{11}{12}W\mu \Rightarrow \mu &= \tfrac{3}{11} \\
\Rightarrow \underline{\underline{\mu} &= 0.27 \text{ (2 sf)}}.
\end{aligned}
$$

26. A non-uniform rod AB, of mass 5 kg and length 4 m, rests with one end A on rough horizontal ground. The centre of mass of the rod is d metres from A. The rod is held in limiting equilibrium at an angle θ to the horizontal by a force $\mathbf{P}$, which acts in a direction perpendicular to the rod at B, as shown in Figure 24.

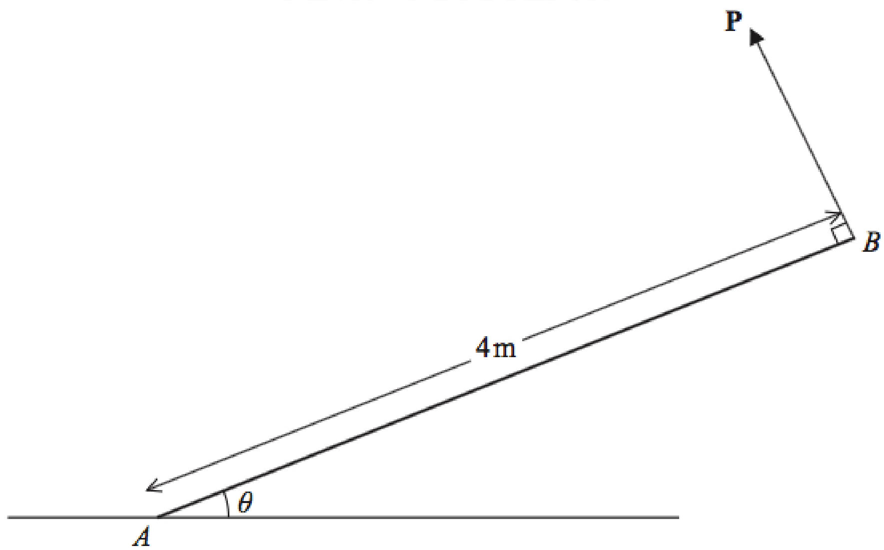


Figure 24: a non-uniform rod AB of mass 5 kg and length 4 m

The line of action of **P** lies in the same vertical plane as the rod.

(a) Find, in terms of d, g, and θ, (8)

(i) the magnitude of the vertical component of the force exerted on the rod by the ground,

Solution

A: F_A N be the frictional force and R_A N be the normal reaction.
B: Let X N and Y N be the horizontal and vertical components of **P**.

$$\begin{aligned} R(\updownarrow):&\quad R_A + P\cos\theta = 5g \\ R(\leftrightarrow):&\quad F_A = P\sin\theta \\ \text{Limiting equilibrium}:&\quad F_A = \tfrac{1}{2}R_A \\ \text{Moments about } A:&\quad d(5g\cos\theta) = 4P. \end{aligned}$$

$$\begin{aligned} & d(5g\cos\theta) = 4P \\ \Rightarrow\ & P = \tfrac{5dg\cos\theta}{4} \\ \Rightarrow\ & R_A = 5g - (\cos\theta \times \tfrac{5dg\cos\theta}{4}) \\ \Rightarrow\ & R_A = 5g - \tfrac{5dg\cos^2\theta}{4} \\ \Rightarrow\ & R_A = \underline{\underline{\tfrac{5g(4-d\cos^2\theta)}{4}}}. \end{aligned}$$

(ii) the magnitude of the friction force acting on the rod at A.

Solution

$$F_A = \frac{5dg\cos\theta}{4} \times \sin\theta = \underline{\underline{\frac{5dg\sin\theta\cos\theta}{4}}}.$$

Given that $\tan\theta = \frac{5}{12}$ and that the coefficient of friction between the rod and the ground is $\frac{1}{2}$,

(b) find the value of d. (4)

Solution

$$\tan\theta = \frac{5}{12} \Rightarrow \sin\theta = \frac{5}{13} \text{ and } \cos\theta = \frac{12}{13}.$$

Now,

$$\begin{aligned}
& \frac{5dg\sin\theta\cos\theta}{4} = \frac{1}{2} \times \frac{5g(4-d\cos^2\theta)}{4} \\
\Rightarrow\ & 2(5d\sin\theta\cos\theta) = 5(4 - d\cos^2\theta) \\
\Rightarrow\ & \frac{600d}{169} = 5(4 - \frac{144d}{169}) \\
\Rightarrow\ & \frac{120d}{169} = 4 - \frac{144d}{169} \\
\Rightarrow\ & \frac{264d}{169} = 4 \\
\Rightarrow\ & d = \frac{169}{66} \\
\Rightarrow\ & \underline{\underline{d = 2.6 \text{ m (2 sf)}}}.
\end{aligned}$$

27. A uniform rod AB, of mass 5 kg and length 8 m, has its end B resting on rough horizontal ground. The rod is held in limiting equilibrium at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$, by a rope attached to the rod at C. The distance $AC = 1$ m. The rope is in the same vertical plane as the rod. The angle between the rope and the rod is β and the tension in the rope is T newtons, as shown in Figure 25.

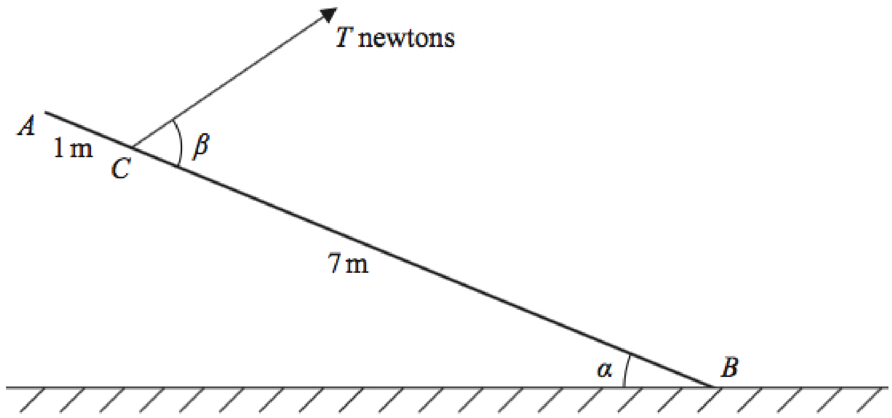


Figure 25: a uniform rod AB of mass 5 kg and length 8 m

The coefficient of friction between the rod and the ground is $\frac{2}{3}$. The vertical component of the force exerted on the rod at B by the ground is R newtons.

(a) Find the value of R. (6)

Solution

B: F N be the frictional force and R N be the normal reaction.

$$\begin{aligned} R(\updownarrow): \quad & R + T\sin(\beta - \alpha) = 5g \\ R(\leftrightarrow): \quad & F = T\cos(\beta - \alpha) \\ \text{Limiting equilibrium}: \quad & F = \tfrac{2}{3}R \\ \text{Moments about } C: \quad & (5g)(3\cos\alpha) + F(7\sin\alpha) = R(7\cos\alpha). \end{aligned}$$

Now,

$$\tan\alpha = \tfrac{3}{4} \Rightarrow \sin\alpha = \tfrac{3}{5} \text{ and } \cos\alpha = \tfrac{4}{5}.$$

Finally,

$$\begin{aligned} & (5g)(3\cos\alpha) + F(7\sin\alpha) = R(7\cos\alpha) \\ \Rightarrow\quad & 15g + \tfrac{2}{3}R(7\tan\alpha) = 7R \\ \Rightarrow\quad & 15g + \tfrac{7}{2}R = 7R \\ \Rightarrow\quad & \tfrac{7}{2}R = 15g \\ \Rightarrow\quad & R = \tfrac{30}{7}g \\ \Rightarrow\quad & \underline{\underline{R = 42 \text{ N}}}. \end{aligned}$$

(b) Find the size of angle β. (5)

Solution

Now,

$$\tan\alpha = \tfrac{3}{4} \Rightarrow \sin\alpha = \tfrac{3}{5},$$
$$F = \tfrac{2}{3} \times 42 = 28,$$

and

$$\begin{aligned}\tan(\beta-\alpha) = \frac{T\sin(\beta-\alpha)}{T\cos(\beta-\alpha)} &\Rightarrow \tan(\beta-\alpha) = \frac{5g-42}{28} \\ &\Rightarrow \beta-\alpha = 14.036\,243\,47 \text{ (FCD)} \\ &\Rightarrow \beta = 50.906\,141\,11 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\beta = 51^\circ \text{ (2 sf)}}}.\end{aligned}$$